Entwistle Mathematics

Year 12 Advanced Mock Exam

SOLUTIONS

General Instructions:

- Reading time – 10 minutes
- Working time – 3 hours
- Write using a black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper.
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section.

Section II – 90 marks

- Attempt Questions 1-30
- Allow about 2 hours and 45 minutes for this section.

Section I

10 marks
Attempt Questions 1–-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–-10.

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1 Consider the line $ax + by + c = 0$ which is neither horizontal or vertical.

What is the gradient of the line?

(A) $-\frac{b}{a}$

(B) $-\frac{a}{b}$

(C) $\frac{a}{b}$

(D) $\frac{b}{a}$

Solution:
(B): Make y the subject so that $y = -\frac{a}{b}x - \frac{c}{b}$ to obtain a gradient of $-\frac{a}{b}$.

2 What is the derivative of $\sin(2^x)$?

(A) $\ln 2 \cos(2^x)$

(B) $\ln 2 \cdot 2^x \cos(2^x)$

(C) $2^x \cos(2^x)$

(D) $\frac{1}{\ln 2} 2^x \cos(2^x)$

Solution:
(B): Use chain rule to get

$$\frac{d}{dx}\left(\sin(2^x)\right) = \frac{d}{dx}\left(2^x\right)\cos(2^x) = \ln 2 \cdot 2^x \cos(2^x)$$

3 What is the period of the function $f(x) = 3\tan(3x)$

(A) $\dfrac{\pi}{3}$

(B) $\dfrac{2\pi}{3}$

(C) π

(D) 2π

Solution:
(A): The period of $y = \tan x$ is π. Replacing x with $3x$ corresponds to a horizontal contraction by a factor of 3. Hence the period is $\dfrac{\pi}{3}$.

4 What is the value of $\displaystyle\int_{-2}^{3} |x-1|\,dx$

(A) $\dfrac{11}{2}$

(B) $\dfrac{13}{2}$

(C) $\dfrac{15}{2}$

(D) $\dfrac{17}{2}$

Solution:
(B): Students can either split the integral and then use definition of absolute value:

$$\int_{-1}^{3} |x-1|\,dx = \int_{-2}^{1} 1 - x\,dx + \int_{1}^{3} x - 1\,dx = \cdots$$

Or alternatively just add the two triangles to obtain $\dfrac{9}{2} + \dfrac{4}{2} = \dfrac{13}{2}$.

5 Suppose X is a normally distributed random variable with mean 10 and some standard deviation σ. If $P(X > 15) = 0.16$, them which of the following is the correct expression for $P(X > -5|X < 0)$?

(A) 0.15

(B) 0.84

(C) 0.94

(D) 0.95

Solution:
(C): If $\mu = 10$ and $P(X > 10 + 5) = 0.16$ then we can conclude that $\sigma = 5$. Hence

$$\begin{aligned} P(X > -5|X < 0) &= \frac{P(-5 < X < 0)}{P(X < 0)} \\ &= \frac{P(-3 < Z < -2)}{P(Z < -2)} \\ &= \frac{\frac{1}{2}(0.997 - 0.95)}{\frac{1}{2}(1 - 0.95)} \\ &= 0.94 \end{aligned}$$

6 The composite function $h(x)$ is defined by $f(g(x))$ where

$$f(x) = \frac{1}{x^2 - 1} \text{ and } g(x) = \sqrt{4 - x}.$$

What is the domain of the composite function?

(A) $x \in (-\infty, 4]$

(B) $x \in (-\infty, 3) \cup (3, 4]$

(C) $x \in (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

(D) $x \in (-\infty, 4) \cup (4, \infty)$

Solution:
(B): For $g(x)$ to exist, we require $4 - x \geq 0$ so $x \leq 4$. For $f(g(x))$ to exists, we require $g(x)^2 \neq 1$, so $4 - x \neq 1$ which means $x \neq 3$. Hence the domain is $x \leq 4$ where $x \neq 3$.

7 Alice plays 2 rounds of some game where the probability of winning each round is p. If you either win or lose on each round and Alice has a 50% chance of winning at least one round.

What is the probability that she wins exactly one round?

(A) $\frac{1}{4}$

(B) $\frac{3}{4}$

(C) $\sqrt{2}-1$

(D) $\frac{1}{2}\left(\sqrt{2}-1\right)$

Solution:
(C): First we have that $1-(1-p)^2 = \frac{1}{2}$. Rearrange for p to obtain $p = 1-\frac{1}{\sqrt{2}}$. Then the probability that she wins exactly one round is

$$2p(1-p) = 2\left(1-\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} = 2\left(\frac{1}{\sqrt{2}}-\frac{1}{2}\right) = \sqrt{2}-1.$$

8 For some angle θ it is given that $\sin\theta < 0$ and $\sec\theta > 0$. Which of the following is guaranteed to be true?

(A) $\cot\theta > 0$

(B) $\operatorname{cosec}\theta\cot\theta < 0$

(C) $\sec\theta + \tan\theta > 0$

(D) $\operatorname{cosec}\theta + \tan\theta > 0$

Solution:
(C): If $\sin\theta < 0$ then θ is in third or fourth quadrant. If $\sec\theta > 0$ then $\cos\theta > 0$ and so θ must be in the fourth quadrant.

(A) is not correct since $\cot\theta < 0$ in fourth quadrant.

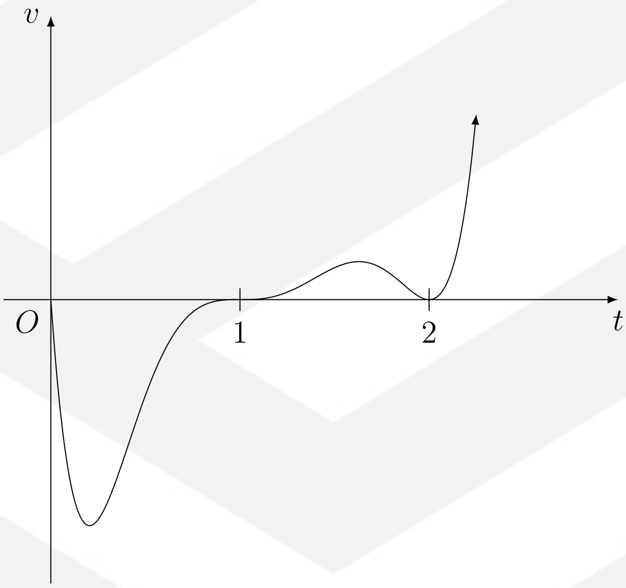
(B) is not correct since $\operatorname{cosec}\theta < 0$ and $\cot\theta < 0$ so $\operatorname{cosec}\theta\cot\theta > 0$.

(C) simplifies to $\dfrac{1+\sin\theta}{\cos\theta}$ which is positive since $1+\sin\theta > 0$ since $-1 < \sin\theta < 0$ and $\cos\theta > 0$. Hence (C) is correct.

(D) is not correct since $\operatorname{cosec}\theta$ and $\tan\theta$ are both negative.

9 A particle is moving along a straight line.

The graph shows the velocity, v, of the particle for time $t \geq 0$.



How many times does the particle change direction?

(A) 1

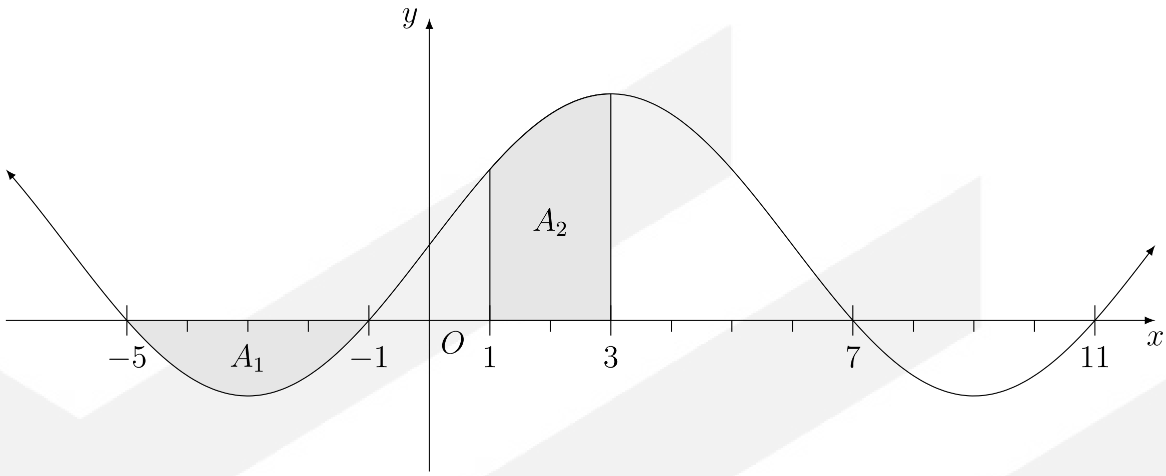
(B) 2

(C) 3

(D) 4

Solution:
(A): Remember that this is the *velocity-time* graph. So the particle changes direction whenever the velocity transitions from negative to positive (or from positive to negative). This happens at $t = 1$ only.

10 The diagram shows the graph of $y = f(x)$ with intercepts at $x = -5, -1, 7$ and 11. It is given that $f(x)$ is some function of the form $A\sin(Bx + C) + D$ for constants A, B, C and D.



The areas A_1 and A_2 are given by 2 and 3 respectively.

It is also known that $\displaystyle\int_{-1}^{11} f(x)\,dx = 8$.

What is the value of $\displaystyle\int_{-5}^{5} f(x)\,dx$?

(A) 2

(B) 4

(C) 6

(D) 8

Solution:

(C): If $\displaystyle\int_{-1}^{11} f(x)\,dx = 8$ then we can conclude that $\displaystyle\int_{-1}^{7} f(x)\,dx = 8 + 2 = 10$.

By symmetry we also have that $\displaystyle\int_{3}^{5} f(x)\,dx = 3$.

Putting this all together gives

$$\int_{-5}^{5} f(x)\,dx = -2 + 2 + 3 + 3 = 6.$$

Section II

89 marks
Attempt Questions 11–-30
Allow about 2 hours and 45 minutes for this section

Question 11 (2 marks)

For what values of x is the function $f(x) = x^3 - 4x^2 + 2x - 1$ concave up? **2**

Solution:

$$\begin{aligned} f(x) &= x^3 - 4x^2 + 2x - 1 \\ f'(x) &= 3x^2 - 8x + 2 \\ f''(x) &= 6x - 8 \end{aligned}$$

Solving $f''(x) > 0$ gives $x > \dfrac{4}{3}$.

Question 12 (3 marks)

Find the equation of the tangent to the curve $y = 2\sqrt{7 - 3x}$ at the point where $y = 2$. **3**

Leave your answer in the form $y = ax + b$.

Solution:
When $y = 2$, $2\sqrt{7-3x} = 2$ gives $x = 2$. Then $y' = 2 \cdot \dfrac{-3}{2\sqrt{7-3x}} = -\dfrac{3}{7-3x} = -3$ at $x = 2$.

Now use point-gradient formula:

$$\begin{aligned} y - 2 &= -3(x - 2) \\ y &= -3(x-2) + 2 \\ &= -3x + 8 \end{aligned}$$

Question 13 (3 marks)

Find the values of θ, where $0 \leq \theta \leq 2\pi$, such that $\sec\left(2\theta - \frac{\pi}{6}\right) = \sqrt{2}$. **3**

Solution:
First adjust the domain: $-\frac{\pi}{6} \leq 2\theta - \frac{\pi}{6} \leq \frac{23\pi}{6}$ then

$$\sec\left(2\theta - \frac{\pi}{6}\right) = \sqrt{2}$$
$$\cos\left(2\theta - \frac{\pi}{6}\right) = \frac{1}{\sqrt{2}}$$

which gives solution in first and fourth quadrants:

$$2\theta - \frac{\pi}{6} = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}$$
$$2\theta = \frac{5\pi}{12}, \frac{23\pi}{12}, \frac{29\pi}{12}, \frac{47\pi}{12}$$
$$\theta = \frac{5\pi}{24}, \frac{23\pi}{24}, \frac{29\pi}{24}, \frac{47\pi}{24}$$

Question 14 (4 marks)

The first, third and ninth term of an arithmetic sequence is given by $1, x$ and y respectively.

(a) Show that $y = 4x - 3$. **2**

Solution:
Suppose the common difference is d. Then $x - 1 = 2d$ and $y - 1 = 8d$. This gives

$$y - 1 = 4(x - 1)$$
$$y - 1 = 4x - 4$$
$$y = 4x - 3$$

(b) Given that $1, x$ and y are distinct numbers that are consecutive terms of a geometric sequence, determine the values of x and y. **2**

Solution:
The common ratio is x, so we must have $x^2 = y$. Substituting for $y = 4x - 3$ gives

$$x^2 = 4x - 3$$
$$x^2 - 4x + 3 = 0$$
$$(x - 3)(x - 1) = 0$$

to obtain $x = 1$ or $x = 3$. However if $x = 1$ then the terms would be $1, 1$ and 1 which are not distinct, hence we must have that $x = 3$. If $x = 3$ then $y = 4 \times 3 - 3 = 9$.

Question 15 (5 marks)

To investigate the association between winning soccer teams and the number of fouls the team committed over the season, an analyst used a linear regression line by collecting the number of goals scored by a particular team over a season of 10 matches and the number of fouls committed over this period.

Using x as the number of fouls and y as the number of goals scored, the equation of the least-squares regression line was found in the form $y = Ax + B$.

It was concluded that on average, for every 4 fouls committed that the number of goals scored would increase by 1.

The total number of goals scored over the season was 15 and the fouls counted were: $1, 4, 5, 3, 0, 2, 7, 6, 10, 3$.

(a) It is known that $(\bar{x}, \bar{y})$ will be a point on the regression line. **3**

Using this fact, along with other information, find the equation of the regression line.

Solution:
We have been told that on average, for every 4 fouls committed that the number of goals will increase by 1 - this must mean that the gradient is 0.25. Now we calculate $\bar{x}$ and $\bar{y}$:

If the total number of goals scores was 15 then we have that $\bar{y} = \dfrac{15}{10} = 1.5$.

Then we can see that $\bar{x} = \dfrac{1+4+5+3+0+2+7+6+10+3}{10} = 4.1$.

Substitute for $\bar{x}, \bar{y}$ and A into the line $y = Ax + B$ to get $1.5 = 0.25 \times 4.1 + B$ to obtain $B = 0.475$.

Hence $y = 0.25x + 0.475$.

Question 15 continues on page 11

(b) The analyst wants to predict the number of fouls committed by the team if 5 goals were scored. **2**

They does this by letting $y = 5$ and then solving for x.

Find **two** issues with this method. One of these methods should refer to the method behind least squares regression.

Solution:

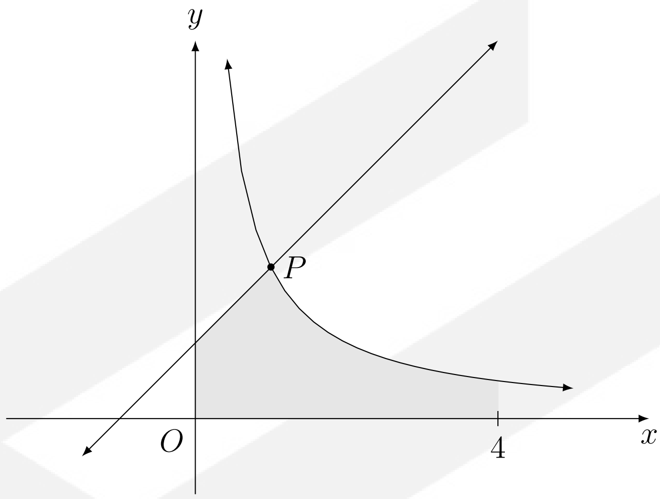
Reason 1: If we do this, then $5 = 0.25x + 0.475$ and rearrange to get $x = 14.1$ goals. If the goals in an entire season was only 15 then it is likely that we are extrapolating from the data set - the linear pattern could change outside of the values that we observed.

Reason 2: This isn't the original line of best fit for x since least squares regressions minimises the square distances from the y-values to the line. It isn't valid to assume that the line has the same minimisation for predicting x values given y values.

NOTE: Random reasons like "not enough data values to do linear regression" etc are not enough to earn marks here - your reason must be specific to what the analyst did (trying to predict x instead of y / extrapolating).

Question 16 (4 marks)

The graphs of $y = \dfrac{2}{x}$ and $y = x + 1$ intersect at the point P as shown in the diagram below. **4**



Find the area of the region bounded by the two curves, the y-axis and the line $x = 4$.

Solution:

First we find P by solving $\dfrac{2}{x} = x + 1$ to obtain

$$\begin{aligned} 2 &= x^2 + x \\ x^2 + x - 2 &= 0 \\ (x+2)(x-1) &= 0 \end{aligned}$$

where $x = 1$ since P is in the first quadrant.

Then the area is given by

$$\begin{aligned} \int_0^1 x + 1\,dx + \int_1^4 \frac{2}{x}\,dx &= \left[\frac{x^2}{2} + x\right]_0^1 + [2\ln x]_1^4 \\ &= 2\ln 4 + \frac{3}{2} \end{aligned}$$

Question 17 (4 marks)

The table below gives the future value of an annuity of \$1 at the end of each period for various periods and interest rates.

Table of Future Value Interest Factors								
	Interest rate per period							
Number of Periods	0.25%	0.30%	0.35%	0.40%	0.45%	0.50%	0.55%	0.60%
53	56.5961	57.3530	58.1230	58.9063	59.7033	60.5141	61.3391	62.1785
54	57.7376	58.5250	59.3264	60.1419	60.9719	61.8167	62.6765	63.5516
55	58.8819	59.7006	60.5340	61.3825	62.2463	63.1258	64.0212	64.9329
56	60.0291	60.8797	61.7459	62.6280	63.5264	64.4414	65.3733	66.3225
57	61.1792	62.0624	62.9620	63.8786	64.8123	65.7636	66.7329	67.7204
58	62.3322	63.2485	64.1824	65.1341	66.1040	67.0924	68.0999	69.1267
59	63.4880	64.4383	65.4070	66.3946	67.4014	68.4279	69.4744	70.5415
60	64.6467	65.6316	66.6359	67.6602	68.7047	69.7700	70.8565	71.9647
61	65.8083	66.8285	67.8692	68.9308	70.0139	71.1189	72.2463	73.3965
62	66.9729	68.0290	69.1067	70.2065	71.3290	72.4745	73.6436	74.8369
63	68.1403	69.2331	70.3486	71.4874	72.6499	73.8368	75.0487	76.2859
64	69.3106	70.4408	71.5948	72.7733	73.9769	75.2060	76.4614	77.7436
65	70.4839	71.6521	72.8454	74.0644	75.3098	76.5821	77.8820	79.2101
66	71.6601	72.8670	74.1004	75.3607	76.6487	77.9650	79.3103	80.6854

A particular bank has an account with an introductory interest rate of 5.4% p.a. for the first year that drops back down to 3% p.a. after the first year. The bank compounds the interest on a monthly basis.

Alice invests 100 at the end of each month into an account earning an interest rate of 3% p.a compounded monthly for 6 years.

(a) Explain why the table cannot be used to find the value of Alice's investment after one year. **1**

Solution:
The monthly interest is 0.25% and the number of periods is 12. However the table starts from 53 onwards so we cannot use the table.

Question 17 continues on page 14

(b) Suppose P is the amount in the account at the end of the first year. **1**

Show that P is \$1230.15.

Solution:

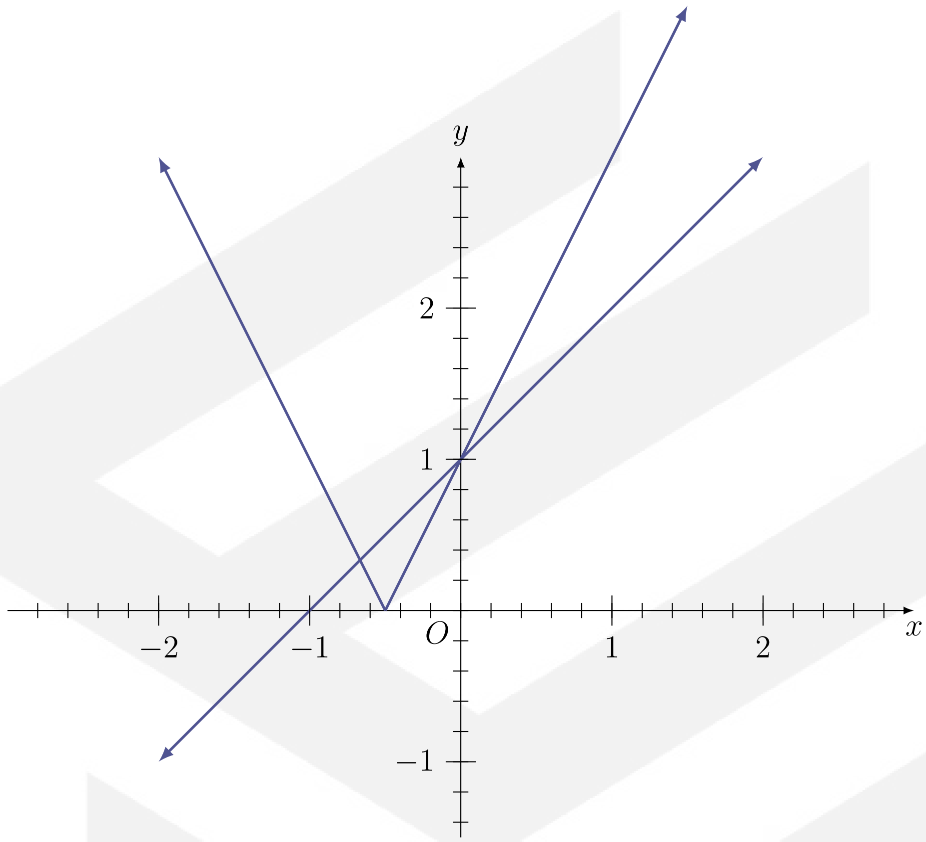
$$\begin{aligned} P &= 100 \times (1 + 1.0045 + \cdots 1.0045^{11}) \\ &= 100 \times \frac{1.0045^{12} - 1}{0.0045} \\ &= 1230.15 \end{aligned}$$

(c) Using the table, along with (b), find the future value of Alice's investment. **2**

Solution:
$1230.15 \times 1.0025^{60} + 100 \times 64.6467 = 7893.63$

Question 18 (4 marks)

(a) Sketch the graphs of the functions $f(x) = x + 1$ and $g(x) = |2x + 1|$ on the axes below, showing x-intercepts. **2**



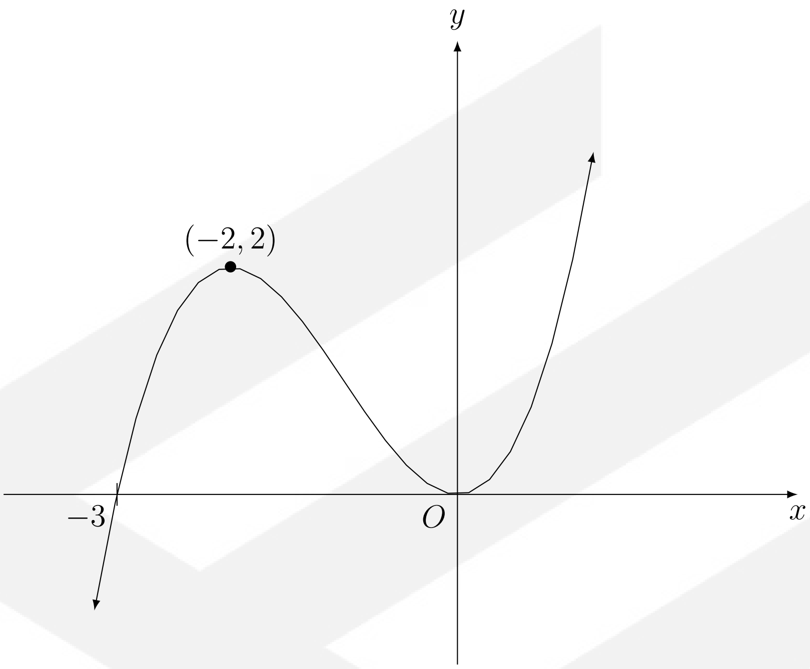
(b) Hence or otherwise, solve the inequality $|2x + 1| \leq x + 1$. **2**

Solution:
Solve the intersection of the two graphs first. They clearly intersect at $x = 0$. Then they also intersect on the left branch of $y = |2x + 1|$, so we solve $-2x - 1 = x + 1$ for $x = -\dfrac{2}{3}$.

Then graphically we have that $-\dfrac{2}{3} \leq x \leq 0$.

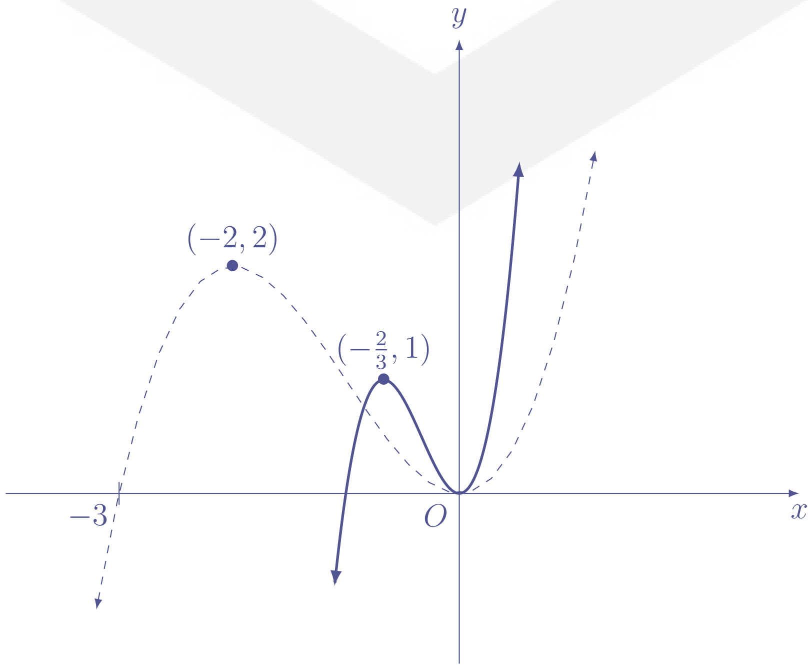
Question 19 (3 marks)

Consider the graph of $y = f(x)$ as shown in the diagram below.



(a) Sketch the graph of $y = \frac{1}{2}f(3x)$, showing the x-intercepts and the coordinate of the turning points. **2**

Solution:



(b) The coordinate $P(a, b)$ is on the final graph of the transformed curve. **1**

Determine the coordinates, in terms of a and b, of the *original* point that corresponds to P under the transformation in (a).

Solution:
To 'un-do' the transformations applied, we dilate horizontally by a factor of 3 and dilate vertically by a factor of 2. The original point would have been $(3a, 2b)$.

To check we could use the point $\left(-\dfrac{2}{3}, 1\right)$ on our graph to make sure the original point is $(-2, 2)$.

Question 20 (4 marks)

Some relation undergoes the following sequence of transformations: **4**

- Shifted to the right by 1 unit,
- dilated horizontally by a factor of 2 and
- contracted vertically by a factor of 3.

The resulting curve is the relation given by $(x-8)^2 + 36y^2 = 16$.
Show that the original relation was a circle and state its centre and radius. **Solution:**

Reverse the transformations:
Dilate vertically by factor of 3 by replacing y with $\dfrac{y}{3}$:

$$\begin{aligned}(x-8)^2 + 36\left(\frac{y}{3}\right)^2 &= 16\\ (x-8)^2 + 4y^2 &= 16\end{aligned}$$

Contract horizontally by a factor of 2 by replacing x with $2x$:

$$\begin{aligned}(2x-8)^2 + 4y^2 &= 16\\ 4(x-4)^2 + 4y^2 &= 16\\ (x-4)^2 + y^2 &= 4\end{aligned}$$

Shift to the left by 1 unit by replacing x with $x+1$:

$$\begin{aligned}(x+1-4)^2 + y^2 &= 4\\ (x-3)^2 + y^2 &= 4\end{aligned}$$

This is a circle centred at $(3, 0)$ with radius 2.

Question 20 continues on page 18

Note: Students are strongly advised to do each transformation in one step - students who attempt to do all three will most likely make an error. Hugh reckons these sorts of "un-do" the transformation will be more common in future exams. If the HSC wants to make it challenging, they will give a *relation* instead of a function which will confuse students depending on how they were taught this topic.

Question 21 (4 marks)

Find the integral $\displaystyle\int_0^{\frac{\pi}{2}} \sin x\sqrt{1+\cos x}\,dx$. **4**

Solution:
Use the formula

$$\int f'(x)\,[f(x)]^n\ dx = \frac{1}{n+1}\,[f(x)]^{n+1} + C$$

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin x\sqrt{1+\cos x}\,dx &= -\int_0^{\frac{\pi}{2}} -\sin x\,(1+\cos x)^{\frac{1}{2}}\ dx\\ &= -\frac{2}{3}\left[(1+\cos x)^{\frac{3}{2}}\right]_0^{\frac{\pi}{2}}\\ &= \frac{2}{3}(2\sqrt{2}-1)\end{aligned}$$

Question 22 (4 marks)

At a harbour, high tide is 12 metres and low tide is 2 metres. It takes 4 hours to go from low tide to high tide. Initially at 2am, it is at low tide. Suppose the height of the tide at a harbour follows the function $x(t) = A\cos(Bt + C) + D$ for some constants A, B, C and D and t is the number of hours after 2am.

(a) Deduce that a suitable displacement time equation can be given by **2**

$$x = 7 - 5\cos\left(\frac{t\pi}{4}\right).$$

Solution:
If high tide is 12 and low tide is 2 then the centre is $\dfrac{12+2}{2} = 7$ (so $D = 7$) and the amplitude is $\dfrac{12-2}{2} = 5$ (so $A = -5$). The period is 8 since it takes 4 hours to go from low to high then back to low. So $\dfrac{2\pi}{B} = 8$ which gives $B = \dfrac{\pi}{4}$. Since the particle starts at low tide, we have $C = 0$.

(b) A ship needs at least at least 5 metres of water in order to safely enter the harbour. **2**

Find the latest time, correct to the nearest minute, that the ship may leave the harbour after entering at the earliest possible time.

Solution:
We solve $x = 5$ for *second* solution (students may draw a diagram to see this). Solving $x = 5$ gives

$$\begin{aligned}
7 - 5\cos\left(\frac{t\pi}{4}\right) &= 5 \\
\cos\left(\frac{t\pi}{4}\right) &= \frac{2}{5} \\
\frac{t\pi}{4} &= \cos^{-1}\left(\frac{2}{5}\right), 2\pi - \cos^{-1}\left(\frac{2}{5}\right) \\
t &= \frac{4}{\pi}\cos^{-1}\left(\frac{2}{5}\right), 8 - \frac{4}{\pi}\cos^{-1}\left(\frac{2}{5}\right)
\end{aligned}$$

The second solution is 6 hours and 31 minutes, hence the time is 8 : 31am.

Question 23 (7 marks)

The continuous random variable X has cdf for some k and $\alpha > 0$,

$$F(x) = \begin{cases} e^{-kx}\left(e^{\alpha x} - 1\right)^2 & : x \geq 0 \\ 0 & : x < 0 \end{cases}$$

(a) Show that $k = 2\alpha$. **3**

Solution:
If $k = 2\alpha$ then $F(x) = (1 - e^{-\alpha x})^2$ where $F(x) \longrightarrow 1$ as $x \longrightarrow 0$ and $F(x) = 0$ when $x = 0$. Moreover $F(x)$ is clearly a non-decreasing function.
Hence $F(x)$ is a valid cdf for $k = 2\alpha$.

We should also demonstrate that $F(x)$ is *not* a valid cdf for $k \neq 2\alpha$:
Note that $F(x) = \left(e^{x\left(\alpha - \frac{k}{2}\right)} - e^{-\frac{k}{2}x}\right)^2$.

As $x \longrightarrow \infty$, we require $F(x) \longrightarrow 1$.

If $\alpha > \dfrac{k}{2}$, then as $x \longrightarrow \infty$ we would have $F(x) \longrightarrow \infty$.

If $\alpha < \dfrac{k}{2}$, then as $x \longrightarrow \infty$ we would have $F(x) \longrightarrow 0$.

Clearly $k > 0$, since otherwise $F(x) \longrightarrow \infty$.

Question 23 continues on page 21

Question 23 (continued)

(b) Suppose that the mode of the random variable X is given by m. **3**

Show that $P(X \leq m)$ does not depend on α.

Solution:
We have that $F(x) = (1 - e^{-\alpha x})^2$. Now differentiate to obtain the PDF:

$$f(x) = 2\left(1 - e^{-\alpha x}\right)\alpha e^{-\alpha x} = 2\alpha\left(e^{-\alpha x} - e^{-2\alpha x}\right)$$

Then find the mode by finding stationary points: $f'(x) = 2\alpha^2 e^{-\alpha x}\left(2e^{-\alpha x} - 1\right)$ Solving $f'(x) = 0$ gives $x = 0$ or $\dfrac{\ln 2}{\alpha}$. Clearly $m \neq 0$ since $f(0) = 0$ so $m = \dfrac{\ln 2}{\alpha}$. Now plug this into $F(m)$ to obtain

$$\begin{aligned} P(X \leq m) &= F\left(\frac{\ln 2}{\alpha}\right) \\ &= e^{-2\alpha \cdot \frac{\ln 2}{\alpha}}\left(e^{\alpha \cdot \frac{\ln 2}{\alpha}} - 1\right)^2 \\ &= e^{-2\ln 2}\left(e^{\ln 2} - 1\right)^2 \\ &= \frac{1}{4} \end{aligned}$$

which is obviously independent of α.

(c) Two different values are observed from this distribution. **1**

Determine the probability that exactly one value is above the mode, and the other value is below the mode.

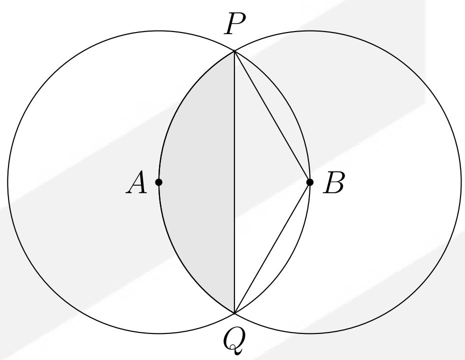
Solution:
Either the first value is above and the second is below OR the first value is below and the second one is above. So the probability is given by

$$2 \times \frac{1}{4} \times \frac{3}{4} = \frac{3}{8}$$

NOTE: Students struggled with (a) (which is fine - its a hard question) but a lot of students did not then attempt (b) and (c) and left 4 marks on the table... Students should always attempt subsequent parts especially since we are told the value of k so there is no barrier to continuing.

Question 24 (6 marks)

Two circles centred at A and B of radius 2cm pass through each others centres as shown in the diagram below. The circles intersect at the points P and Q and the region between the chord PQ and the arc subtended by the second circle is shaded in the diagram below.

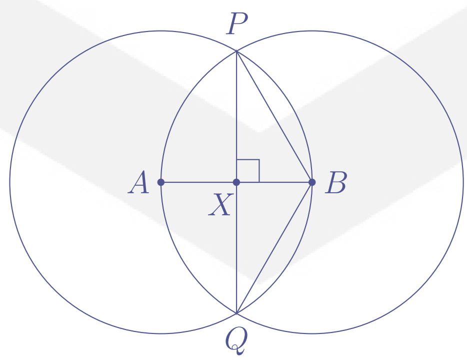


(a) Find the size of $\angle PBQ$. **2**

Solution:

Consider drawing the line AB and PQ to intersect at X. By symmetry we must have that $\angle PXB = \frac{\pi}{2}$ with $XB = \dfrac{1}{2}$ and $PB = 1$.

Hence $\cos(\angle PBX) = \dfrac{\frac{1}{2}}{1} = \dfrac{1}{2}$ so $\angle PBX = \dfrac{\pi}{3}$. Double this to obtain $\angle PBQ = \dfrac{2\pi}{3}$.

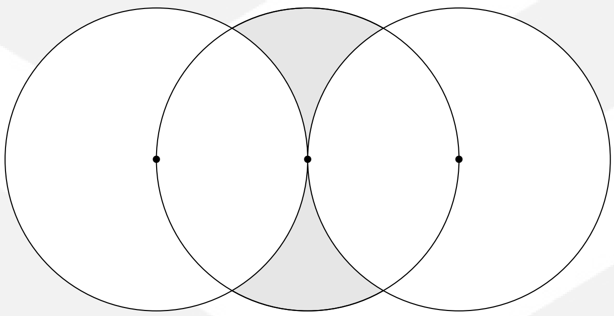


(b) Hence determine the area of the shaded region. **2**

Solution:
Use area of a sector formula $A = \dfrac{1}{2}r^2\theta$ to find the area of the sector from B. Then subtract the area of $\triangle PBQ$ to obtain:

$$\begin{aligned}
\text{Shaded Region} &= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta \\
&= \frac{1}{2}r^2(\theta - \sin\theta) \\
&= \frac{1}{2} \times 4 \times \left[\frac{2\pi}{3} - \sin\left(\frac{2\pi}{3}\right)\right] \\
&= 2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) \\
&= \frac{4\pi - 3\sqrt{3}}{3}
\end{aligned}$$

Now consider three circles of radius 2cm such that the two outermost circles are tangential and the middle circle passes through the centres of the outer circles. The region formed by the intersection of the outer circles with the middle circle is now shaded on the diagram below.



(c) By considering (b), or otherwise, determine the area of the new shaded region. **2**

Solution:
Subtract *four* copies of the region in (b) from the area of the inner-most circle.

We obtain the area:

$$\pi \times 2^2 - 4 \times \frac{4\pi - 3\sqrt{3}}{3} = \frac{4}{3}\left(3\sqrt{3} - \pi\right) \approx 2.73 \text{ (correct to two decimal places)}$$

Question 25 (5 marks)

In a game, two fair dice are rolled and the score is the maximum of the two dice. For example if a 3 and a 5 are rolled, then the score is 5. Let the score be given by the random variable X.

(a) By considering a 6×6 grid of values, or otherwise, construct a probability distribution table for X. **2**

Solution:
We will let students draw that 6×6 grid table themselves! To provide some extra value to the student studiously reading these solutions, lets show you something cool...

We can write:

$$P(X = x) = P(X \leq x) - P(X \leq x - 1).$$

Since $P(X \leq x)$ is equivalent to *both* dice having a value $\leq x$, then this is easy to calculate:

$$P(X = x) = \left(\frac{x}{6}\right)^2 - \left(\frac{x-1}{6}\right)^2 = \frac{2x-1}{36}$$

Then we just populate the probability distribution table:

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

Students should check that their probabilities sum to 1 if they are unsure!

(b) Hence determine the expected value of X. **1**

Solution:

$$\begin{aligned} E(X) &= 1 \times \frac{1}{36} + 2 \times \frac{3}{36} + 3 \times \frac{5}{36} + 4 \times \frac{7}{36} + 5 \times \frac{9}{36} + 6 \times \frac{11}{36} \\ &= \frac{1 + 6 + 15 + 28 + 45 + 66}{36} \\ &= \frac{161}{36} \\ &\approx 4.5 \end{aligned}$$

As a sanity check this should obviously be ≥ 4 since the average of a die is 3.5 and we are receiving the maximum, so intuitively ≥ 4 on average.

Another trick is to use the fact that, for discrete random variables that take positive integer values, that $E(X) = P(X > 0) + P(X > 1) + \cdots + P(X > 5)$ where $P(X > x) = 1 - P(X \leq x) = 1 - \frac{x^2}{36}$ so $E(X) = 6 - \frac{1}{36}\left(1^2 + 2^2 + 3^2 + 4^2 + 5^2\right)$

(c) Hugh is offered to play this game up to two times. If he doesn't like the score in the first round, then he can choose to re-roll to try and get a better score. **2**

Hugh will re-roll if the score is less than $E(X)$. Determine his expected final score with this strategy.

Solution:
Clearly the strategy is that Hugh re-rolls if he gets 4 or less on the first roll, and will not re-roll if he got 5 or higher. When Hugh re-rolls, he will get $E(X)$ on average for his second roll. Hence his expected score is given by

$$\begin{aligned} E(S) &= E(X) \times \frac{1}{36} + E(X) \times \frac{3}{36} + E(X) \times \frac{5}{36} + E(X) \times \frac{7}{36} + 5 \times \frac{9}{36} + 6 \times \frac{11}{36} \\ &= \frac{\frac{161}{36} \times (1+3+5+7) + 45 + 66}{36} \\ &= \frac{1643}{324} \\ &\approx 5.07 \end{aligned}$$

Question 26 (5 marks)

Alice inherits \$100 000 and invests it into an account earning interest at a rate of 1.2% per annum, compounded monthly. Each month, immediately after interest has been paid, Alice withdraws \$1000. **5**

The amount in the account immediately after the n^{th} withdrawal can be determined using the recurrence relation

$$A_n = A_{n-1}(1.001) - 1000.$$

where $n = 1, 2, 3, \ldots$ and $A_0 = 100\,000$.

In a particular month, Alice finds that she has insufficient funds to take out the usual \$1000 withdrawal and instead just takes out all of the remaining balance of $\$M$.

Determine the value of M correct to the nearest cent.

Solution:
Keep applying the reccurence relation to see that

$$\begin{aligned} A_n &= 100000 \times 1.001^n - 1000\left(1 + 1.001 + \cdots + 1.001^{n-1}\right) \\ &= 100000 \times 1.001^n - 1000 \cdot \frac{1.001^n - 1}{0.001} \\ &\ldots \\ &= 100000\left(10 - 9 \times 1.001^n\right) \end{aligned}$$

Now solve for when $A_n = 0$:

$$\begin{aligned} 100000\left(10 - 9 \times 1.001^n\right) &= 0 \\ 1.001^n &= \frac{10}{9} \\ n\ln(1.001) &= \ln\left(\frac{10}{9}\right) \\ n &= \frac{\ln\left(\frac{10}{9}\right)}{\ln(1.001)} \\ &\approx 105.41\ldots \end{aligned}$$

So after the 105^{th} withdrawal there is a balance of $A_{105} \approx \$412.895$. Then interest will apply for one more month and then we clear out the account.

Hence the last withdrawal is

$$\begin{aligned} M &= A_{105} \times 1.001 \\ &= \$413.31 \text{ (correct to the nearest cent)} \end{aligned}$$

Most students forget this last step.

Question 27 (5 marks)

For this question, you may use the fact that the average of n normally distributed variables with mean μ and variance σ^2 will also follow a normal distribution with mean μ and variance $\frac{\sigma^2}{n}$.

The weight of a potato is assumed to be normally distributed with mean 150 grams and standard deviation of 10 grams.

A grocery store wishes to sell bags of potatoes such that the average weight of each potato is at least 145 grams.

(a) For a bag of four potatoes, determine the probability that the average weight is no less than 145 grams. **2**

Solution:
Let X be the average weight of each potato in the bag. Then the distribution of X is normal with mean 150 and variance $\dfrac{100}{4} = 25$ which is a standard deviation of 5 grams.

Then

$$\begin{aligned} P(X > 145) &= P\left(Z > \frac{145 - 150}{5}\right) \\ &= P(Z > -1) \\ &= 0.68 + 0.16 \\ &= 0.84 \end{aligned}$$

Question 27 continues on page 28

(b) Determine the minimum number of potatoes that should be in the bag to ensure that the probability that the average weight is less than 145 grams is less than 2.5%. **3**

Solution:
If there are n potatoes then the average now follows a normal distribution with mean 150 and variance $\dfrac{100}{n}$ which is a standard deviation of $\dfrac{10}{\sqrt{n}}$. So

$$\begin{aligned} P(X < 145) &= P\left(Z < \frac{145-150}{\frac{10}{\sqrt{n}}}\right) \\ &= P\left(Z < -\frac{\sqrt{n}}{2}\right) \end{aligned}$$

We want this probability to be less than 2.5% so the z-score needs to be < -2, this gives:

$$\begin{aligned} -\frac{\sqrt{n}}{2} &< -2 \\ \frac{\sqrt{n}}{2} &> 2 \\ \sqrt{n} &> 4 \\ n &> 16 \end{aligned}$$

So the minimum number is $n = 17$ (Allow a mark for students who say $n = 16$ since we are using an approximation anyway).

Question 28 (6 marks)

Consider the function $f(x) = x^2e^{-x}$.

(a) Find the coordinates of any stationary points and determine their nature. **2**

Solution:

$$\begin{aligned} f(x) &= x^2e^{-x} \\ f'(x) &= 2xe^{-x} - x^2e^{-x} \\ &= xe^{-x}(2-x) \\ f''(x) &= 2e^{-x} - 2xe^{-x} - 2xe^{-x} + x^2e^{-x} \\ &= \left[(x-2)^2 - 2\right] e^{-x} \end{aligned}$$

Solving $f'(x) = 0$ gives $x = 0$ and $x = 2$ so $(0, 0)$ and $\left(2, \dfrac{4}{e^2}\right)$ are stationary points.
To determine their nature we look at the second derivative.

When $x = 0$, $f''(x) = 2 > 0$ so $(0, 0)$ is a minimum.

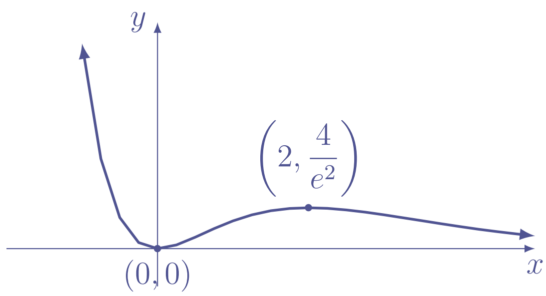
When $x = 2$, $f''(x) = -2e^{-2} < 0$ so $\left(2, \dfrac{4}{e^2}\right)$ is a maximum.

(b) Determine the behaviour of $f(x)$ as $x \to \infty$ and $x \to -\infty$. **2**

Solution:
As $x \to \infty$, $f(x) \to 0$. As $x \to -\infty$, $f(x) \to \infty$.

(c) Sketch the curve, labelling the turning points and intercepts. **2**

Solution:



Question 29 (5 marks)

Alice wishes to sell an item and has an unknown reserve price X for which she will sell the item if she receives an offer that is above the reserve price (and will not sell if the offer is below the reserve price).

The reserve price will be no lower than \$500 and no higher than \$1000. It is known that the distribution of the reserve price scales linearly from 0 at 500 to most likely at 1000 such that the probability distribution of the random variable X is of the form

$$f(x) = \begin{cases} ax + b, & 500 \leq x \leq 1000 \\ 0, & \text{else} \end{cases}$$

for constants a and b.

(a) Find the value of a and b. **2**

Solution:
We need $f(500) = 0$ which gives $500a + b = 0$. But then the area under the distribution must be equal to 1, so this gives $\frac{1}{2} \times 500 \times (1000a + b) = 1$. This gives the two equations:

$$\begin{aligned} 500a + b &= 0 \\ 250(1000a + b) &= 1 \end{aligned}$$

Substitute for $b = -500a$ from the first equation into the second equation to obtain:

$$\begin{aligned} 250(1000a - 500a) &= 1 \\ 125000a &= 1 \\ a &= \frac{1}{125000} \end{aligned}$$

and so $b = -500a = -\frac{1}{250}$.

So $a = \frac{1}{125000}$ and $b = -\frac{1}{250} = -0.004$.

Question 29 continues on page 31

(b) Bob knows that he can sell the item for \$1000 and so will submit an offer of x to Alice and immediately sell the item if he successfully receives it from Alice. **3**

What value of x maximises Bob's expected profit?

Solution:
If Bob offers below Alice's reserve price, then he will not be able to sell it (and have 0 profit).

If Bob offers above Alice's reserve price, then he will get it (and then have $1000 - x$ profit).

Hence Bob's expected profit, in terms of x, is

$$\begin{aligned} f(x) &= P(X > x) \times 0 + P(X \leq x) \times (1000 - x) \\ &= 0 + (1000 - x) \int_{500}^{x} ax + b\, dx \\ &= 0 + \frac{1}{2}(x - 500)(ax + b) \times (1000 - x) \\ &= \frac{1}{250000}(x - 500)^2(1000 - x) \end{aligned}$$

So maximising this:

$$\begin{aligned} f'(x) &= \frac{1}{250000}\left[2(x - 500)(1000 - x) - (x - 500)^2\right] \\ &= \frac{x - 500}{250000}(2500 - 3x) \end{aligned}$$

where $f'(x) = 0$ at $x = 500$ and $x = \dfrac{2500}{3}$.

If Bob offers $x = 500$ then he will definitely NOT get the item since Alice's reserve price is above 500. His profit would then be \$0.

If Bob offers $x = \dfrac{2500}{3}$, then his expected profit is

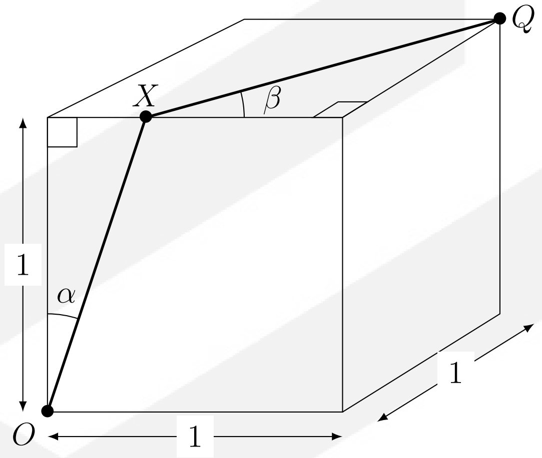
$$f\left(\frac{2500}{3}\right) = \frac{1}{250000}\left(\frac{2500}{3} - 500\right)^2\left(1000 - \frac{2500}{3}\right) \approx \$74.07$$

so Bob should offer about \$833.33 for the item and he will lock in an immediate \$74.07 profit.

It turns out that the value of x is coincidentally the expected value of Alice's belief. Students who just calculated the expected value of Alice's reserve price will only get 1 mark, unless they can explain *why* this ends up happening. Do you think this is true for any probability distribution? For any price that Bob can then sell it?

Question 30 (6 marks)

Consider an ant on the corner of a unit cube. The ant travels at an angle of α to the vertical towards the edge of the cube and then travels at an angle of β to the horizontal to the opposite corner as shown in the diagram below.



Let L be the total distance that the ant travels in this scenario.

(a) Show that the total distance is given by **2**

$$L = \sec\alpha + \sqrt{1 + (1 - \tan\alpha)^2}$$

Solution:
In the first right angle triangle we have that

$$\cos\alpha = \frac{1}{OX} \implies OX = \sec\alpha.$$

If we let the vertex directly above the origin by equal to P then we can see that $PX = \tan\alpha$ and so the base of the seonc triangle is $1 - \tan\alpha$. Using pythag, we then have that

$$XQ = \sqrt{1 + (1 - \tan\alpha)^2}$$

Finally

$$L = OX + XQ = \sec\alpha + \sqrt{1 + (1 - \tan\alpha)^2}.$$

Question 30 continues on page 33

(b) By differentiating the expression in (a) with respect to α, find the value of α that minimises L. **3**

Solution:

$$\begin{aligned}\frac{dL}{d\alpha} &= \sec\alpha\tan\alpha + \frac{-2(1-\tan\alpha)\sec^2\alpha}{2\sqrt{1+(1-\tan\alpha)^2}} \\ &= \sec\alpha\left(\tan\alpha - \frac{(1-\tan\alpha)\sec\alpha}{\sqrt{1+(1-\tan\alpha)^2}}\right)\end{aligned}$$

Solving $\dfrac{dL}{d\alpha} = 0$ gives $\sec\alpha = 0$ (no solutions) or

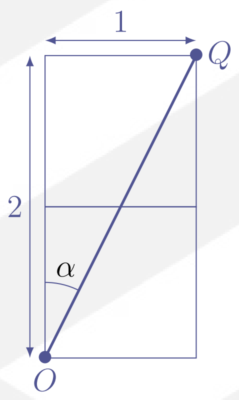
$$\begin{aligned}\tan\alpha &= \frac{(1-\tan\alpha)\sec\alpha}{\sqrt{1+(1-\tan\alpha)^2}} \\ \tan^2\alpha\left(1+(1-\tan\alpha)^2\right) &= (1-\tan\alpha)^2\sec^2\alpha \\ (1-\tan\alpha)^2\left(\sec^2\alpha - \tan^2\alpha\right) &= \tan^2\alpha \\ (1-\tan\alpha)^2 &= \tan^2\alpha \\ \tan^2\alpha - (1-\tan\alpha)^2 &= 0 \\ \tan^2\alpha - 1 + 2\tan\alpha - \tan^2\alpha &= 0 \\ 2\tan\alpha &= 1 \\ \tan\alpha &= \frac{1}{2}\end{aligned}$$

so $\alpha = \tan^{-1}\left(\dfrac{1}{2}\right) \approx 26^\circ$ minimises L. This is clearly the minimum as the maximum would be obtained when travelling the full diagonal to get $L = 1+\sqrt{2}$.

Question 30 continues on page 34

(c) By considering the relationship between α and β, provide a geometric interpretation of the minimising value of α in terms of the net of the cube. **1**

Solution:
Since $\alpha = \beta$, it implies that the path taken was actually a straight line. To see this more clearly, unfold the cube to see the net (or we just only need two squares to demonstrate) then clearly shortest route would be to travel the straight line distance from O to Q as shown:



where it is now obvious to see that $\tan\alpha = \dfrac{1}{2}$

End of Exam