Entwistle Mathematics

Year 12 Advanced Mock Exam

General Instructions:

- Reading time – 10 minutes
- Working time – 3 hours
- Write using a black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper.
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section.

Section II – 90 marks

- Attempt Questions 1-29
- Allow about 2 hours and 45 minutes for this section.

Section I

10 marks
Attempt Questions 1–-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–-10.

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1 What is the solution to $\left|\frac{2-x}{3}\right| \geq 2$

(A) $x > 4$ or $x < -8$

(B) $x > 8$ or $x < -4$

(C) $-8 < x < 4$

(D) $-4 < x < 8$

2 What is the domain of $f(x) = \frac{1}{\sqrt{x^2 - x}}$?

(A) $x < 0$ or $x > 1$

(B) $0 < x < 1$

(C) $x \leq 0$ or $x \geq 1$

(D) $0 \leq x \leq 1$

3 How many consecutive terms of the arithmetic sequence $30, 26, 22, \ldots$ can be taken to give a sum of 120?

(A) 8 or 10

(B) 6 or 10

(C) 6 only

(D) 10 only

4 Let $f(x) = 2\ln(3x)$. Which of the following transformations applied to $f(x)$ would correctly obtain $g(x) = \ln x$?

(A) Dilate horizontally by a factor of 3 and then dilate vertically by a factor of 2.

(B) Contract vertically by a factor of 2 and then contract horizontally by a factor of 3.

(C) Contract vertically by a factor of 2 and then shift upwards by $\ln 3$ units.

(D) Contract vertically by a factor of 2 and then shift downwards by $\ln 3$ units.

5 The point $P(2, -4)$ lies on $y = f(x)$. A sequence of transformations is applied to $y = g(x)$ to obtain

$$g(x) = 2f\left(\frac{4-x}{3}\right) - 1$$

Which of the points on the graph of $g(x)$ correspond to the original point P?

(A) $(-2, -8)$

(B) $\left(\frac{2}{3}, -8\right)$

(C) $(-2, -9)$

(D) $\left(\frac{2}{3}, -9\right)$

6 Suppose a particle moves in a straight line with some displacement, velocity and acceleration functions at time $t > 0$ given by $x(t), v(t)$ and $a(t)$ respectively.

Which of the following options is correct given that particle is moving to the left and slowing down?

(A) $v > 0$ and $a > 0$

(B) $v > 0$ and $a < 0$

(C) $v < 0$ and $a < 0$

(D) $v < 0$ and $a > 0$

7 Every student in the school is allowed to vote for whom they think should be the school captain by submitting a vote in favour of their top preference.

Let G be the group of students who voted and C be the group of students who were nominated. The relation described between the groups G and C is best described:

(A) One-to-one

(B) One-to-many

(C) Many-to-many

(D) Many-to-one

8 Two fair six-sided die is rolled repeatedly until either a 5 or a 7 is rolled.

What is the probability, correct to 2 decimal places, that a 5 is rolled before a 7 is rolled?

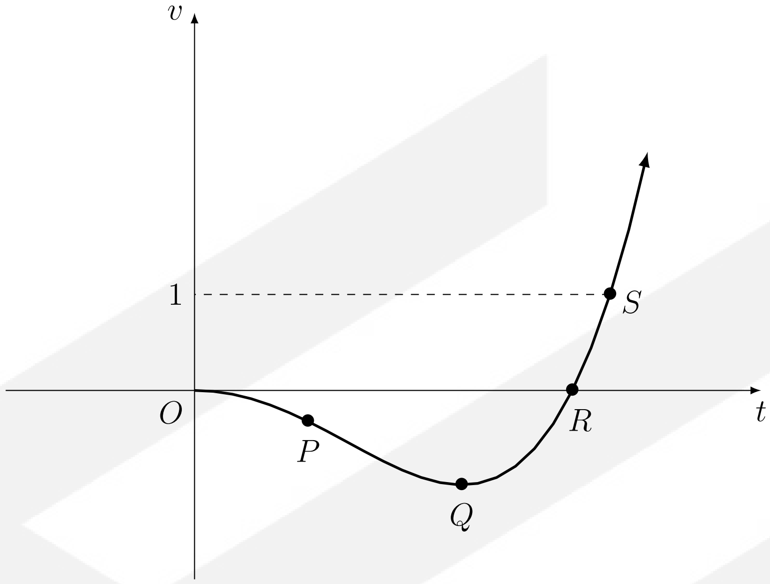
(A) 0.2

(B) 0.25

(C) 0.33

(D) 0.4

9 The following graph is the velocity-time graph for some particle moving in a straight line. The particle has been travelling for some time $T > 0$ corresponding to the t coordinate of one of the points P, Q, R or S which are labelled in the diagram below.



Let the displacement, velocity and acceleration of the particle at time t be given by $x(t), v(t)$ and $a(t)$ respectively.

The following information is known about the particle at time $t = T$:

- $x(T) = 0$ and $x(a) = 1$ for some $0 < a < T$.
- $x(b) > 0$ for all $b > T$.

Which coordinate best describes the current status of the particle?

(A) P

(B) Q

(C) R

(D) S

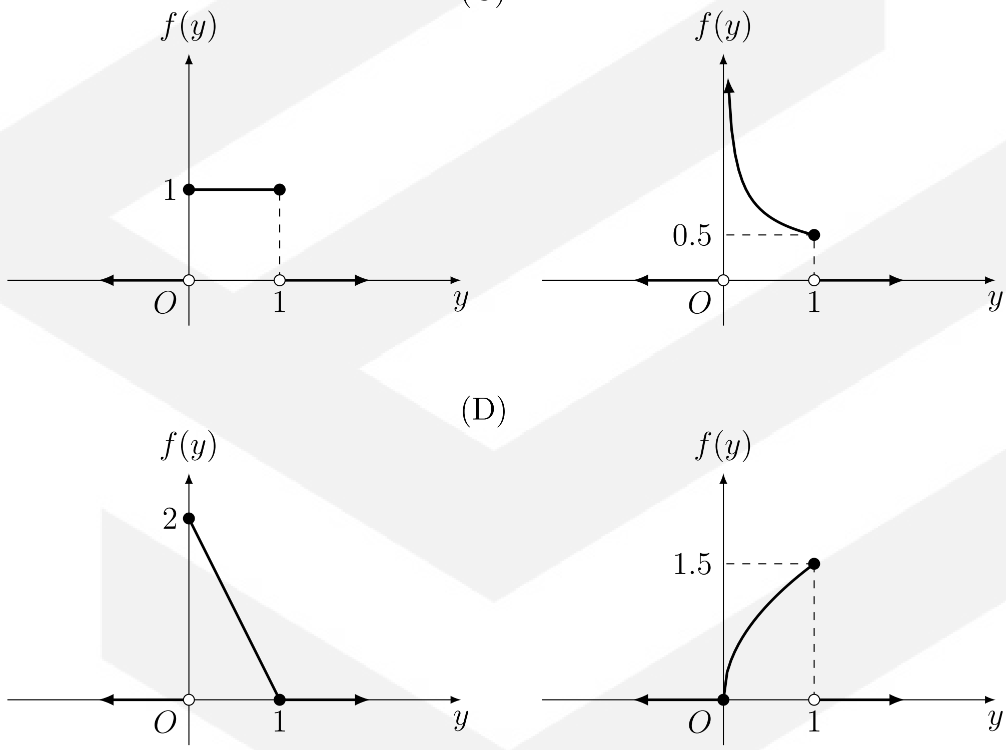
10 Let the continuous random variable X be uniform on the interval $[0, 1]$ with probability density function

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

Let Y be the random variable X^2.

Which of the following graphs best describes the probability density function, $f(y)$, for Y?

(A) (C) (B) (D)



Section II

91 marks
Attempt Questions 11–-29
Allow about 2 hours and 45 minutes for this section

Question 11 (3 marks)

The third and eight term of an arithmetic sequence is given by 20 and 60. Find the first term that is greater than 100. **3**

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Question 12 (2 marks)

Evaluate $\log_{\sqrt{m}}\left(\dfrac{1}{m^3}\right)$ for $m > 0$ as constant that does not depend on m. **2**

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Question 13 (4 marks)

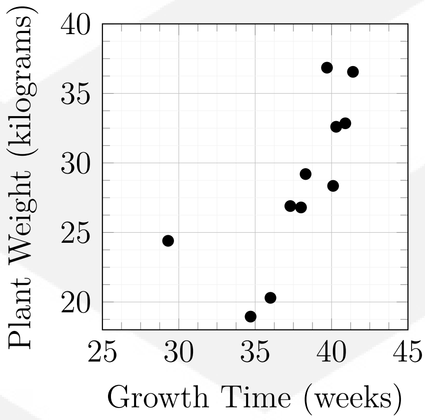
The weight of a particular species of tree was found to follow a linear relationship with the number of weeks it has been growing for after 25 weeks.

The weight and time for several trees from the species was collected and the results are shown in the scatterplot below.

A least squares linear regression was conducted with a Pearson's correlation coefficient of 0.71 and the line of best fit was found to be

$$y = -16.929 + 1.2018x$$

where x is the growth time (in weeks) and y is the weight of the tree (in kilograms).



(a) It is known that $(\bar{x}, \bar{y})$ always lies on the regression line. **2**

Given that $\bar{x}$ is 37.82, find $\bar{y}$ correct to two decimal places.

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(b) Based on the regression line, what is the average daily growth in kilograms? **2**

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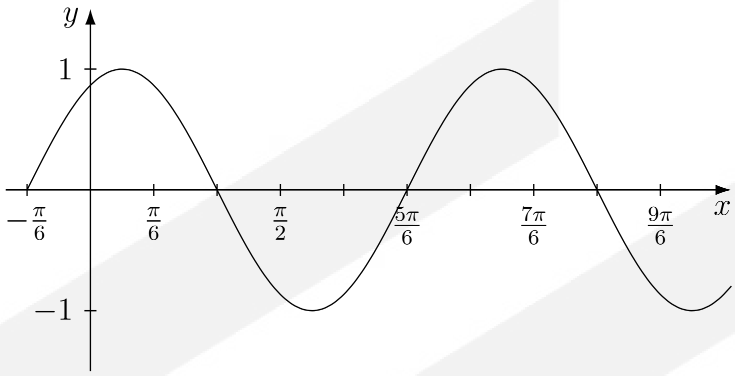
Question 14 (2 marks)

The graph below is part of a curve with equation $y = \sin(ax - b)$ where $a > 0$, $0 < b \leq \pi$. **2**
Use the information on the diagram to find the values of a and b.



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Question 15 (4 marks)

Solve the equation $7 \times 2^{x-3} = 2^{2x-3} + 1$. **4**

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Question 16 (4 marks)

A table of future value interest factors for an annuity of \$1, paid at the end of each time period, is shown below. Two of the entries are missing and are referred to as x and y.

r \ N	*Interest rate per period as a decimal*					
	0.002	0.006	0.020	0.024	0.060	0.240
10	10.09048	10.27437	10.94972	11.15211	13.18079	31.64344
20	20.38460	21.18211	24.29737	25.28909	36.78559	303.60062
30	30.88646	x	40.56808	43.20983	79.05819	2640.91639
40	41.60026	y	60.40198	65.92708	154.76197	22728.80260

(a) Use the table to explain why $y = 10.27437 + x\,(1.006)^{10}$. **1**

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It is now given that $x = 32.76227$ and $y = 45.05630$ **(Do NOT prove this).**

(b) Alice regularly deposits \$500 at the end of each quarter into an account earning interest of 2.4% per annum (compounded quarterly). **2**

Determine the balance in Alice's account after 10 years.

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(c) Hence determine the *present value* of Alice's investment. **1**

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Question 17 (4 marks)

(a) Express x^3 in the form $x^3 = x\,(a(x^2+1) - b)$ for constants a and b. **1**

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(b) Hence find $\displaystyle\int x^3\sqrt{x^2+1}\,dx$ **3**

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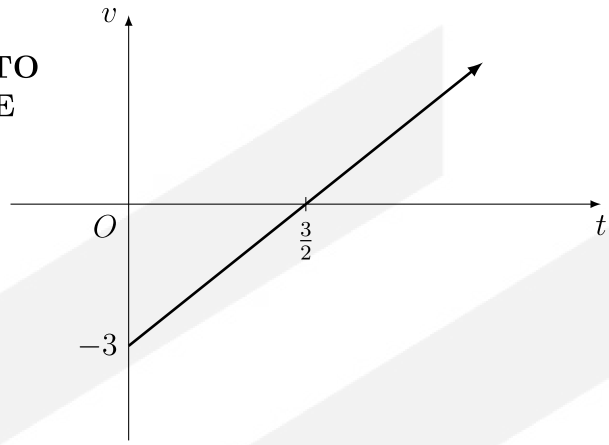
Question 18 (4 marks)

Consider the following velocity-time graph:



The particle is initially 1 metre to the right of the origin.

(a) Find an expression in terms of t for x, the displacement of the particle at time t. **2**

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(b) Find the distance travelled by the particle in the first 2 seconds. **2**

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Question 19 (5 marks)

An infinite geometric series has a second term of -4 and a limiting sum of 9.

(a) Calculate the common ratio. **3**

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(b) Determine the value of n needed for the sum of the first n terms to exceed half of the limiting sum. **2**

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Question 20 (4 marks)

Consider the equation $2\operatorname{cosec}(\pi + x) + \cos\left(\frac{\pi}{2} - x\right) = 1$

(a) Express the equation in the form **2**

$$a\sin^2 x + b\sin x + c = 0$$

for constants a, b and c.

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(b) Hence, or otherwise, solve the equation on the domain $-\pi \le x \le \pi$. **2**

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Question 21 (5 marks)

Alice writes down the following probability distribution table to model some discrete random variable X.

x	0	1	2
$P(X = x)$	$0.5 - p^2$	p^2	0.5

(a) Find the range of values of p for which this is a valid probability distribution. **2**

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(b) Determine the largest possible value for $E(X)$ and find the corresponding variance. **3**

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Question 22 (8 marks)

Consider the graph of $y = 3x^2 - x^3$ in the domain $x \in [-3, 3]$.

(a) Find the stationary points and determine their nature. **3**

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(b) Find the x-coordinate of the point of inflection and prove that it is one. **2**

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Question 22 continues on page 18

Question 22 (continued)

(c) Sketch the curve, labelling all features. **2**

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(d) State the global maximum and minimum. **1**

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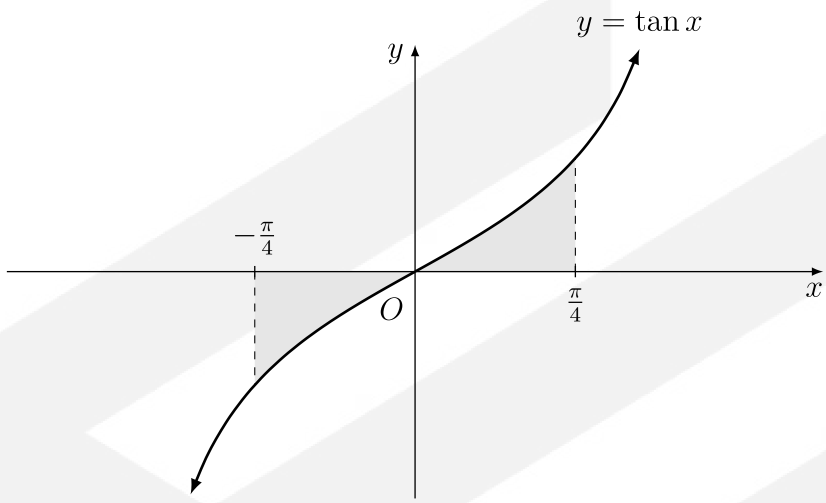
Question 23 (6 marks)

The diagram below shows the graph of $y = \tan x$ where the region bounded by the graph and the lines $x = -\dfrac{\pi}{4}$ and $x = \dfrac{\pi}{4}$.



(a) Show that the area of the shaded region is $\ln 2$. **2**

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Question 23 continues on page 20

(b) Use the trapezoidal rule with 5 function values to approximate $\int_0^{\frac{\pi}{4}} \tan x \, dx$ correct to two decimal places. **2**

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(You may NOT use a calculator for the rest of this question)

(c) Deduce that $\ln 2 \approx 0.7$ and determine whether this is an over-approximation or an under-approximation of the true value of $\ln(2)$, with justification. **2**

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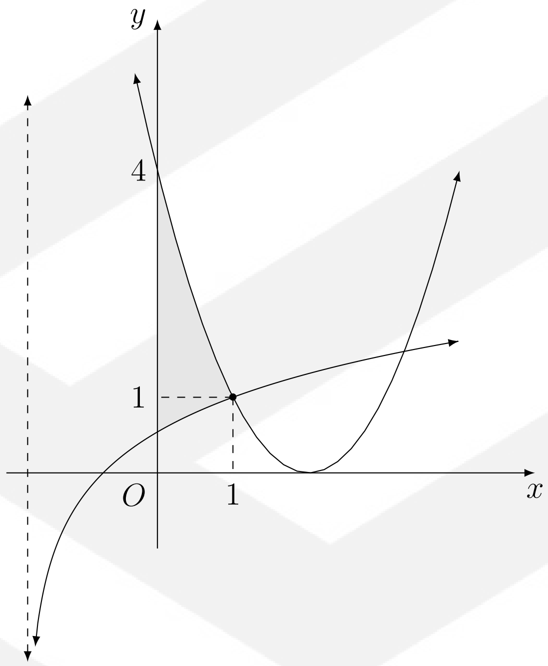
Question 24 (5 marks)

The graphs of the functions $y = \ln(x + k)$ and $y = (x - 2)^2$ are shown on the diagram below. The positive constant $k > 0$ has been chosen such that the two curves intersect at the point $(1, 1)$.

The region bounded between the two curves and the y-axis is shaded in the diagram.



(a) Determine the value of the constant k. **1**

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Question 24 continues on page 22

Question 24 (continued)

(b) Find the shaded area bounded by the two curves and the y-axis is given by **3**

$$A = \frac{7}{3} + (e-1)\left(\ln(e-1) - 1\right).$$

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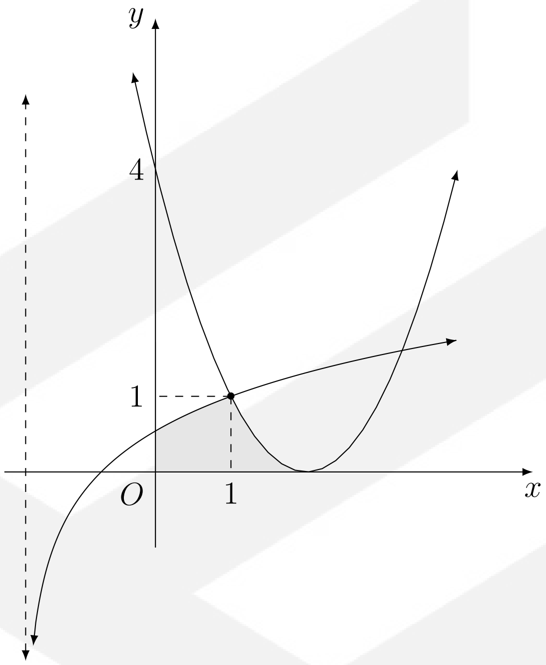
Question 24 continues on page 23

Question 24 (continued)

The region bounded by the curves and the x-axis is now shown on the same diagram below.



(c) Using (b), or otherwise, find the *new* shaded region bounded by the two curves and the x-axis correct to 2 decimal places. **1**

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Question 25 (9 marks)

A population P of insects is given by $P = Ae^{kt}$, where A and k are constants and t is the time in years. An initial population of 500 insects has doubled after 3 years.

(a) Show that $k = \dfrac{\ln 2}{3}$. **2**

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(b) At what rate is the population of insects increasing after 1 year? **2**

Give your answer to the nearest whole number of insects per year.

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Question 25 continues on page 25

Question 25 (continued)

After 5 years the population is deemed too large. At the end of each subsequent year, 500 insects are exterminated from the population in an attempt to control the population.

(c) Let I_n be the number of insects remaining *immediately after* the n^{th} extermination. **2**

Explain why $I_n = \sqrt[3]{2} \times I_{n-1} - 500$.

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(d) How many exterminations are required to destroy this insect population? **3**

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Question 26 (3 marks)

A large cube is formed by taking 27 smaller equally sizes white cubes that are assembled into a $3 \times 3 \times 3$ arrangement. The surface area is then painted green, so that the 27 smaller cubes have a combination of white and green faces.

One of the cubes is chosen at random and tossed onto a table so that only one face is not visible.

(a) What is the probability that the smaller cube has at least one green face? **1**

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(b) Given that all the visible faces were white, what is the probability that the smaller cube only has white faces? **2**

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Question 27 (5 marks)

The waiting time for a bus to arrive at a certain bus-stop is given by a random variable T whose probability density function is given by

$$f(t) = \begin{cases} ke^{-kt}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Alice says to her friend Bob: *"If I have already waited some amount of time for the bus to arrive, then it is more likely that I don't need to wait as long now."*

(a) Show that $P(T \geq t) = e^{-kt}$ for $t > 0$. **2**

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Bob says that Alice is wrong, that it doesn't matter how long she has already been waiting and calculates $P(T \geq a + b | T \geq b)$ to analyse his argument.

(b) Calculate $P(T \geq a + b | T \geq b)$ and discuss whether or not it proves or disproves Bob's claim. **3**

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Question 28 (5 marks)

Consider the functions $f(x) = \dfrac{1}{\sqrt{x^2+1}-5}$, $g(x) = \sqrt{x-1}$ and $h(x) = \dfrac{1}{\sqrt{x}-5}$.

(a) Find the domain of $g(x)$. **1**

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(b) Show that the composite function $f(g(x)) = \dfrac{1}{\sqrt{x}-5}$ where x is in the domain of $f(g(x))$. **2**

Deduce that the domain of $y = h(x)$ is **not** equal to the domain of $y = f(g(x))$.

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Question 28 continues on page 29

(c) The range of $y = h(x)$ is $y \in \left(-\infty, -\frac{1}{5}\right] \cup (0, \infty)$. **(Do NOT prove this)**. **2**

The range of $y = f(g(x))$ is the same as the range of $y = h(x)$ **except** for some interval $(a, b]$.

Find the values of a and b.

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Question 29 (9 marks)

Alice plays a game with two identical boxes. In each turn of the game, she can either "deposit" or "empty". If she "deposits", \$1 is placed at random into one of the boxes (with each box equally likely). If she "takes", then she will receive the contents of one of the boxes at random.

The game consists of 100 turns. Note that Alice does not see how much money is in either box, neither does she see how much she has received whenever she "withdraws" until the end of the 100 turns.

(a) Suppose Alice deposits for 99 turns and then withdraws on the last turn. **1**

What is the expected number of coins that she receives?

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Alice decides to deposit for the first x turns, where $1 \leq x \leq 99$ and then withdraw for the rest of the game.

(b) Explain why the probability that Alice ends up withdrawing from both boxes is given by **2**

$$1 - \left(\frac{1}{2}\right)^{99-x}$$

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Question 29 continues on page 31

(c) Deduce that the expected number of coins withdrawn is given by **2**

$$f(x) = x\left[1 - \left(\frac{1}{2}\right)^{100-x}\right]$$

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(d) Show that $f'(x) = 1 - \left(\frac{1}{2}\right)^{100-x}(1 - x\ln 2)$. **2**

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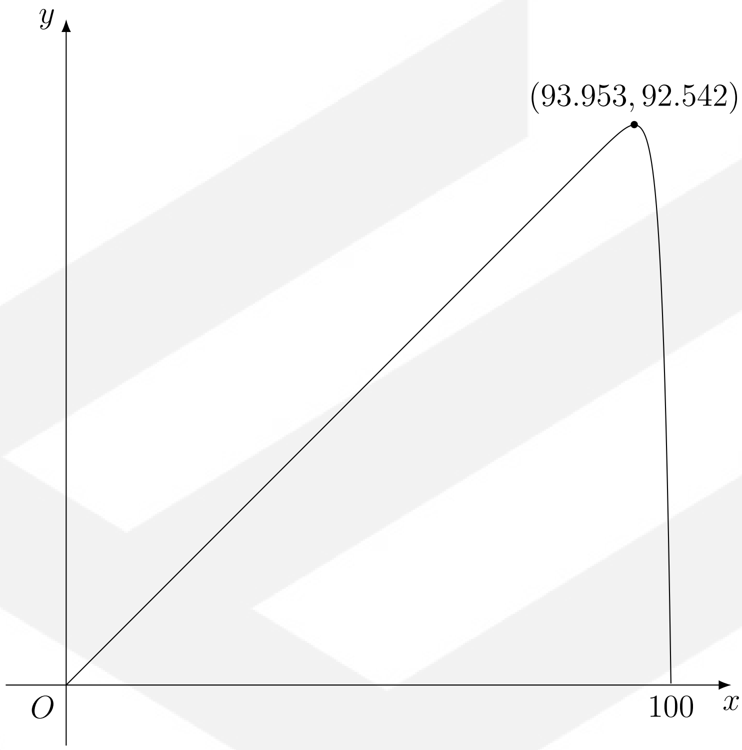
Question 29 continues on page 32

(e) The graph of $y = f(x)$ from (c) is shown below where the solution to $f'(x) = 0$ is approximately $x = 93.953$ correct to three decimal places. **(Do NOT prove this)** **2**



Determine the optimal value of x that Alice should choose to maximise the expected earnings. Briefly explain a potential modification to this strategy that might improve Alice's expected winnings (you do not need to provide any mathematical detail).

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End of Exam