Entwistle Mathematics

Year 12 Extension 1 Mock Exam

General Instructions:

- Reading time – 10 minutes
- Working time – 2 hours
- Write using a black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper.
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section.

Section II – 60 marks

- Attempt Questions 1-14
- Allow about 1 hours and 45 minutes for this section.

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1−10.

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1 Which expression is equal to $\int \cos^2 2x\,dx$?

(A) $\frac{1}{8}\sin^3 2x + C$

(B) $\frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right) + C$

(C) $\frac{1}{2}\left(x - \frac{1}{4}\sin 4x\right) + C$

(D) $-2\sin 2x + C$

2 Which of the following functions does *not* have an inverse function?

(A) $f(x) = x + \ln x$

(B) $f(x) = x + \sin x$

(C) $f(x) = x + \frac{1}{x}$

(D) $f(x) = \tan(\tan^{-1} x)$

3 A bag has twelve red balls, seven green balls, and b black balls.

Suppose that 15 is the smallest number of balls to take from the bag to guarantee that you have six balls of the same colour. Determine the number of black balls in the bag.

(A) 6

(B) 5

(C) 4

(D) 3

4 What is the domain and range of the function $f(x) = \cos^{-1}\left(\cos^{-1}(x)\right)$?

(A) Domain: $-1 \leq x \leq 1$. Range: $0 \leq y \leq \pi$

(B) Domain: $\cos(1) \leq x \leq 1$. Range: $0 \leq y \leq \pi$

(C) Domain: $\cos(1) \leq x \leq 1$. Range: $0 \leq y \leq \frac{\pi}{2}$

(D) Domain: $-1 \leq x \leq 1$. Range: $0 \leq y \leq \frac{\pi}{2}$

5 A polynomial leaves a remainder of $2x+1$ when divided by x^2-1 and a remainder of k when divided by $x+1$ for some constant k.

Which of the following is a correct expression for the remainder when divided by $x-1$?

(A) $-3k$

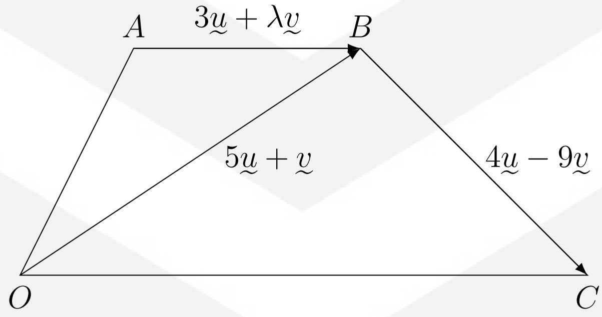
(B) $-k$

(C) k

(D) $3k$

6 Consider the trapezium $OABC$ where AB is parallel to OC.

The trapezium is formed by the vectors as labeled on the diagram, where λ is some constant.



Which of the following is the correct value of λ?

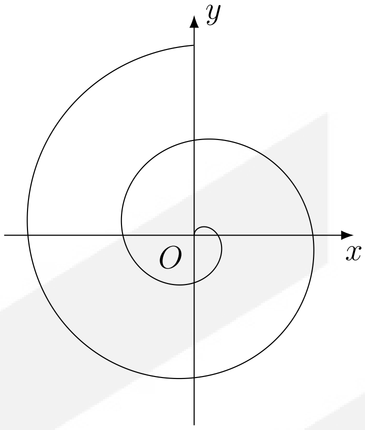
(A) $-\dfrac{3}{8}$

(B) $-\dfrac{8}{3}$

(C) $\dfrac{8}{3}$

(D) $\dfrac{3}{8}$

7 The point $P(t)$ is defined parametrically by some parameter t to form the following curve:



Which of the following best describes a possible pair of parametric equations for x and y?

(A) $x = t\sin t$ and $y = t\cos t$ for $0 \leq t \leq 4\pi$

(B) $x = t\cos t$ and $y = -t\sin t$ for $0 \leq t \leq 4\pi$

(C) $x = t\cos t$ and $y = t\sin t$ for $0 \leq t \leq 4\pi$

(D) $x = -t\cos t$ and $y = t\sin t$ for $0 \leq t \leq 4\pi$

8 Suppose a tank currently has 100L of water with 10kg of salt. A salt solution with a concentration 0.2kg/L is poured into the tank at a rate of 3L/min whilst the evenly mixed solution in the tank drains out at 4L/min.

Which of the following is the correct differential equation for the amount of salt, $S(t)$, in the tank at time t?

(A) $\dfrac{dS}{dt} = \dfrac{1}{25}\left(\dfrac{1}{5} - S\right)$

(B) $\dfrac{dS}{dt} = \dfrac{4}{5}\left(1 - \dfrac{5S}{100 - t}\right)$

(C) $\dfrac{dS}{dt} = \dfrac{4}{5}\left(1 + \dfrac{20S}{3(100 - t)}\right)$

(D) $\dfrac{dS}{dt} = \dfrac{4}{5}\left(1 + \dfrac{20S}{4(100 - t)}\right)$

9 Consider the composite function $p(x) = f(g(x))$ and suppose that the inverse function of $y = p(x)$ exists and is given by $y = q(x)$.

The following information about the functions is known:

x	1	2	3
$f(x)$	-2	-1	0
$g(x)$	4	3	2
$f'(x)$	2	3	4
$g'(x)$	-3	-2	-1

where the domain and range of $p(x)$ and $q(x)$ is over all real numbers.

Which of the following is the correct value of $q'(-1)$?

(A) $-\dfrac{1}{3}$

(B) $\dfrac{1}{3}$

(C) $\dfrac{1}{4}$

(D) $-\dfrac{1}{4}$

10 Hugh draws a card without replacement from a deck of 52 cards and places them into a discard pile until the Ace of Spades is drawn.

What is the probability that exactly one Jack, Queen and King appear in the discard pile?

(A) $\dfrac{1}{2197}$

(B) $\dfrac{16}{715}$

(C) $\dfrac{1}{4}$

(D) $\dfrac{6}{13}$

Section II

60 marks
Attempt Questions 11–-14
Allow about 1 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available. In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booket.

(a) For the vectors $\underset{\sim}{u} = \underset{\sim}{i} + \underset{\sim}{j}$ and the vector $\underset{\sim}{v} = 3\underset{\sim}{i} - 2\underset{\sim}{j}$ evaluate $\underset{\sim}{u} \cdot (\underset{\sim}{u} - 2\underset{\sim}{v})$. **2**

(b) Solve $\dfrac{x}{3-x} \geq 2$. **3**

(c) Evaluate $\displaystyle\int_0^1 4x(2x-1)^4\,dx$ using the substitution $u = 2x - 1$. **3**

(d) Find the coefficient of x^2 in the expansion of $(1+2x)\left(1 - \dfrac{x}{2}\right)^5$. **2**

(e) A non-biased coin is flipped 100 times. Let X be the random variable representing the number of heads obtained.

(i) Find the expected value and standard deviation for X. **2**

(ii) Use a normal approximation to determine the probability that at least 60 heads are flipped. **1**

(f) How many distinct arrangements are there of the word COMMUNICATE where the C's are on both ends and the letter A is not next to the letter T? **2**

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booket.

(a) Consider the polynomial $P(x) = x^4 - 3x^3 - 6x^2 + ax + b$ where a and b are integers. **3**

Given that there is a triple root, determine the possible values of a and b.

(b) Consider the two vectors

$$\underset{\sim}{u} = \left(\frac{1}{2}p^2 + q^2 + 3\right)\underset{\sim}{i} + 4\underset{\sim}{j} \quad \text{and} \quad \underset{\sim}{v} = (p+q)\underset{\sim}{i} + 2\underset{\sim}{j}.$$

Suppose that the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are parallel.

(i) Show that $(p-2)^2 + 2(q-1)^2 = 0$. **2**

(ii) Hence, or otherwise, deduce if it is possible for $\underset{\sim}{u}$ and $\underset{\sim}{v}$ to be parallel. **1**

If it is possible, state the values of p and q for which this is true.

(c) (i) Express $3\sin x + \sqrt{3}\cos x$ in the form $R\sin(x+\alpha)$ for $R > 0$ and some acute angle α. **2**

(ii) Hence solve for the smallest positive value of k for which **3**

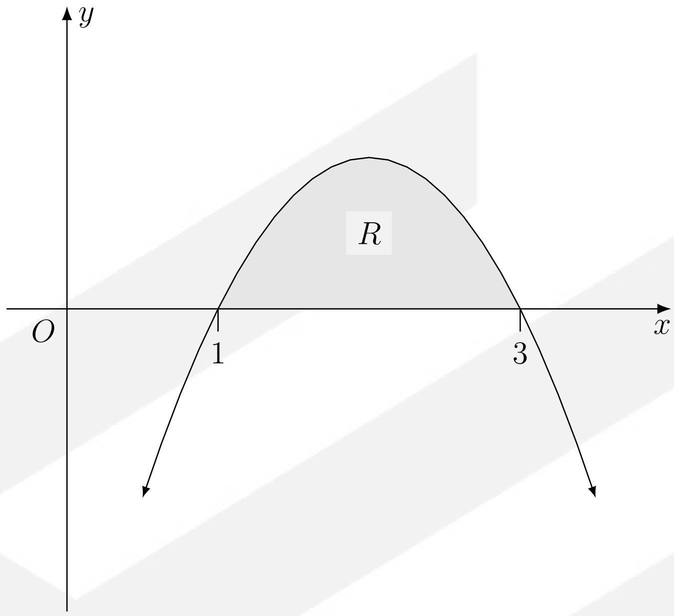
$$\int_0^k \sqrt{3}\sin x - 3\cos x \, dx = 0$$

(d) Prove by mathematical induction that $2^{3n+2} + 5^{n+1}$ is always a multiple of 3 for any positive integer $n \geq 1$. **4**

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booket.

(a) The diagram below shows the graph of the parabola $y = 4x - x^2 - 3$. **3**



The region R bounded by the parabola and the x-axis is rotated about the y-axis to form a solid.

Determine the exact value of the volume for the solid formed.

(b) Let $P(x)$ and $Q(x)$ be monic cubic polynomials. **2**

The polynomial $P(x)$ has roots α, β and γ and the polynomial $Q(x)$ has roots $\alpha^2, \beta^2, \gamma^2$.

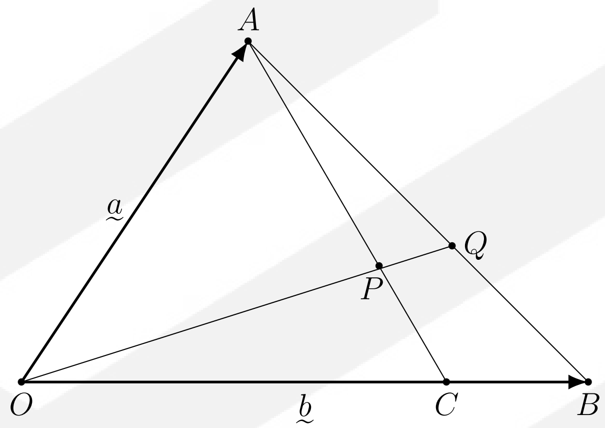
Show that $Q(x) = -P(-\sqrt{x})P(\sqrt{x})$.

Question 13 continues on page 9

Question 13 (continued)

(c) Consider the triangle OAB formed by the vectors $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$. The point C lies on OB such that $OC : CB = 3 : 1$ and the point P lies on AC such that $AP : PC = 2 : 1$. The line OP is extended to intersect the line AB at the point Q.

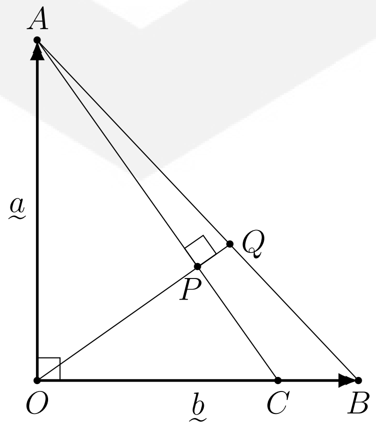
An example of such a triangle with this construction is shown in the diagram below



(i) Show that $\overrightarrow{AP} = -\frac{2}{3}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}$ and $\overrightarrow{PC} = -\frac{1}{3}\underset{\sim}{a} + \frac{1}{4}\underset{\sim}{b}$. **1**

(ii) Determine the ratio $AQ : QB$ in the form $m : n$ where m and n are positive integers in their smallest form. **2**

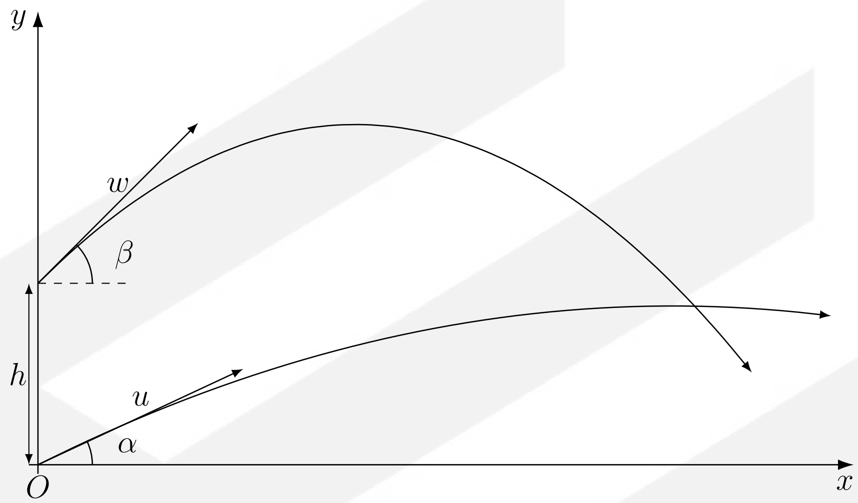
(iii) The diagram below shows the same construction but for the special case where $\angle AOB = \angle APQ = \frac{\pi}{2}$.



By any means necessary, determine the exact value of $\frac{|\underset{\sim}{a}|}{|\underset{\sim}{b}|}$. **2**

Question 13 continues on page 10

(d) Particle A is fired from the origin with initial speed u and initial angle α from the horizontal.



At the same instant, particle B is fired simultaneously from a point h metres directly above particle A. It has initial speed w and initial angle β from the horizontal.

The two particles collide above the horizontal after T seconds.

(i) Show that $T = \dfrac{h\cos\beta}{u\sin(\alpha - \beta)}$ **3**

(ii) Suppose that the particles intersect when A has reached its greatest height. **2**

For any fixed β, determine the angle α that would allow h to be maximised.

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booket.

(a) Consider a continuous random variable X that has a continuous probability density function $f(x)$. Let the cumulative density function be given by $y = F(x)$.

(i) Explain why, on the domain of X, the inverse cumulative density function, $y = F^{-1}(x)$ exists. **1**

(ii) Let U be a random variable that has the uniform distribution **2**

$$f(u) = \begin{cases} 1, & 0 \leq u \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Show that the random variable $F(X)$ has the same distribution as U.

(b) Consider the differential equation

$$\frac{dy}{dx} = (x+y)^2.$$

(i) If the equation is separable, then there are functions $f(x)$ and $g(y)$ so that **2**

$$(x+y)^2 = f(x)g(y).$$

Explain why it must be true that

$$(x+y)^2 = f(x)g(1) \times f(0)g(y) = f(x)g(0) \times f(1)g(y).$$

and hence explain why the equation is not separable.

(ii) Let $u = x + y$ and show that $\dfrac{du}{dx} = 1 + u^2$. **2**

(iii) Hence, or otherwise, find the particular solution to the differential equation $\dfrac{dy}{dx} = x^2 + 2xy + y^2$ that passes through the point $\left(0, \dfrac{\pi}{4}\right)$. **2**

Question 14 continues on page 12

(c) Alice is performing at a local theatre and has a group of $n+1$ friends but only 3 tickets. One of the tickets is for premium seating and the other two are standard seating (but are distinguishable).

Alice gives out the tickets according to two rules:

- The premium ticket will go to the oldest person who gets a ticket.
- The person who gets the premium ticket can't have any more tickets given to them.

(i) Based on the rules, Alice can only either give all of the tickets to three of her friends, or to two of her friends. **2**

Provide a combinatorial argument to show that there are

$$2\binom{n+1}{3}+\binom{n+1}{2}$$

ways to distribute the tickets.

(ii) Explain why the expression in (i) is equivalent to **2**

$$1^2+2^2+3^2+\cdots+n^2$$

ways of distributing the tickets. You **must** provide a combinatorial argument.

(iii) It is known that $\binom{k}{2}+\binom{k+1}{2}=k^2$ **(Do NOT prove this.)** **2**

Use this, along with the previous parts to simplify

$$\binom{n}{2}+\binom{n-1}{2}+\binom{n-2}{2}+\cdots+\binom{2}{2}$$

to the form $\binom{a}{b}$ for a pair of constants a and b.

End of Question 14