Entwistle Mathematics

Year 12 Advanced Mock Exam

General Instructions:

- Reading time – 10 minutes
- Working time – 3 hours
- Write using a black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper.
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section.

Section II – 90 marks

- Attempt Questions 1-30
- Allow about 2 hours and 45 minutes for this section.

Section I

10 marks
Attempt Questions 1–-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–-10.

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1 Consider the line $ax + by + c = 0$ which is neither horizontal or vertical.

What is the gradient of the line?

(A) $-\frac{b}{a}$

(B) $-\frac{a}{b}$

(C) $\frac{a}{b}$

(D) $\frac{b}{a}$

2 What is the derivative of $\sin(2^x)$?

(A) $\ln 2 \cos(2^x)$

(B) $\ln 2 \cdot 2^x \cos(2^x)$

(C) $2^x \cos(2^x)$

(D) $\frac{1}{\ln 2} 2^x \cos(2^x)$

3 What is the period of the function $f(x) = 3\tan(3x)$

(A) $\dfrac{\pi}{3}$

(B) $\dfrac{2\pi}{3}$

(C) π

(D) 2π

4 What is the value of $\displaystyle\int_{-2}^{3} |x-1|\,dx$

(A) $\dfrac{11}{2}$

(B) $\dfrac{13}{2}$

(C) $\dfrac{15}{2}$

(D) $\dfrac{17}{2}$

5 Suppose X is a normally distributed random variable with mean 10 and some standard deviation σ. If $P(X > 15) = 0.16$, them which of the following is the correct expression for $P(X > -5|X < 0)$?

(A) 0.15

(B) 0.84

(C) 0.94

(D) 0.95

6 The composite function $h(x)$ is defined by $f(g(x))$ where

$$f(x) = \frac{1}{x^2 - 1} \text{ and } g(x) = \sqrt{4 - x}.$$

What is the domain of the composite function?

(A) $x \in (-\infty, 4]$

(B) $x \in (-\infty, 3) \cup (3, 4]$

(C) $x \in (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

(D) $x \in (-\infty, 4) \cup (4, \infty)$

7 Alice plays 2 rounds of some game where the probability of winning each round is p. If you either win or lose on each round and Alice has a 50% chance of winning at least one round.

What is the probability that she wins exactly one round?

(A) $\frac{1}{4}$

(B) $\frac{3}{4}$

(C) $\sqrt{2} - 1$

(D) $\frac{1}{2}\left(\sqrt{2} - 1\right)$

8 For some angle θ it is given that $\sin\theta < 0$ and $\sec\theta > 0$. Which of the following is guaranteed to be true?

(A) $\cot\theta > 0$

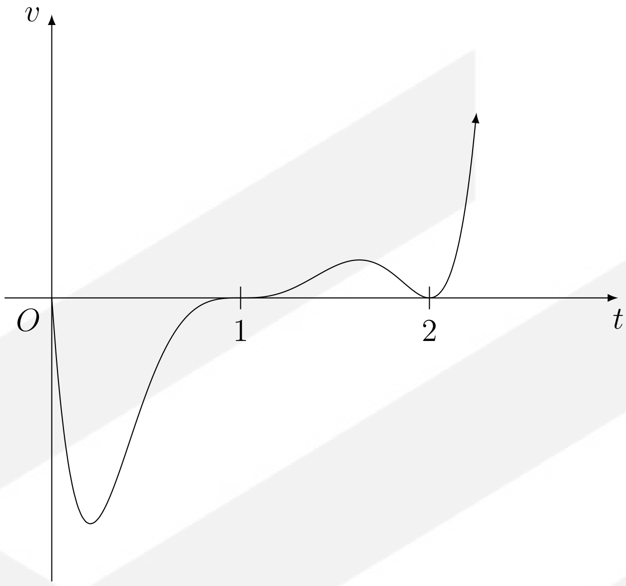
(B) $\operatorname{cosec}\theta \cot\theta < 0$

(C) $\sec\theta + \tan\theta > 0$

(D) $\operatorname{cosec}\theta + \tan\theta > 0$

9 A particle is moving along a straight line.

The graph shows the velocity, v, of the particle for time $t \geq 0$.



How many times does the particle change direction?

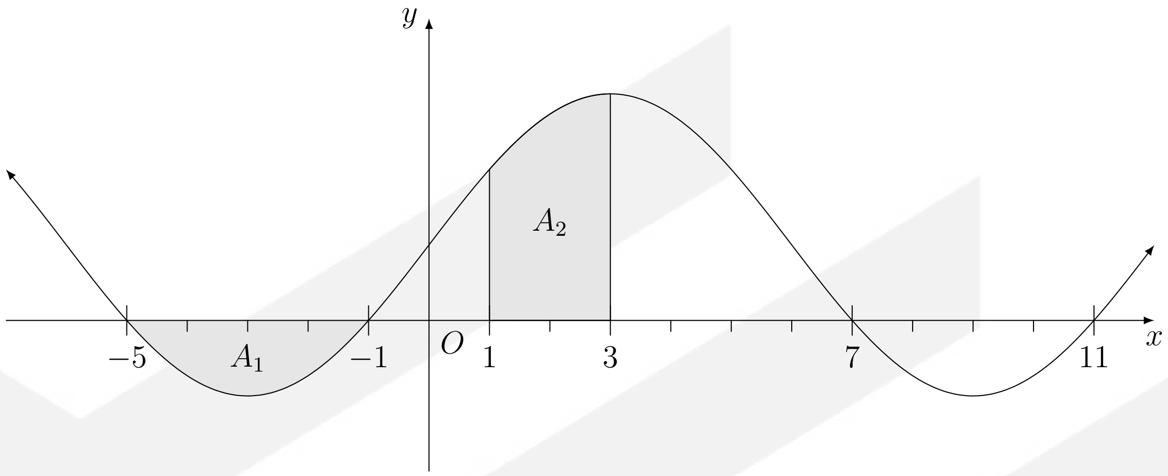
(A) 1

(B) 2

(C) 3

(D) 4

10 The diagram shows the graph of $y = f(x)$ with intercepts at $x = -5, -1, 7$ and 11. It is given that $f(x)$ is some function of the form $A\sin(Bx + C) + D$ for constants A, B, C and D.



The areas A_1 and A_2 are given by 2 and 3 respectively.

It is also known that $\displaystyle\int_{-1}^{11} f(x)\,dx = 8$.

What is the value of $\displaystyle\int_{-5}^{5} f(x)\,dx$?

(A) 2

(B) 4

(C) 6

(D) 8

Section II

90 marks
Attempt Questions 11–-30
Allow about 2 hours and 45 minutes for this section

Question 11 (2 marks)

For what values of x is the function $f(x) = x^3 - 4x^2 + 2x - 1$ concave up? **2**

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Question 12 (3 marks)

Find the equation of the tangent to the curve $y = 2\sqrt{7-3x}$ at the point where $y = 2$. **3**

Leave your answer in the form $y = ax + b$.

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Question 13 (3 marks)

Find the values of θ, where $0 \le \theta \le 2\pi$, such that $\sec\left(2\theta - \frac{\pi}{6}\right) = \sqrt{2}$. **3**

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Question 14 (4 marks)

The first, third and ninth term of an arithmetic sequence is given by $1, x$ and y respectively.

(a) Show that $y = 4x - 3$. **2**

(b) Given that $1, x$ and y are distinct numbers that are consecutive terms of a geometric sequence, determine the values of x and y. **2**

Question 15 (5 marks)

To investigate the association between winning soccer teams and the number of fouls the team committed over the season, an analyst used a linear regression line by collecting the number of goals scored by a particular team over a season of 10 matches and the number of fouls committed over this period.

Using x as the number of fouls and y as the number of goals scored, the equation of the least-squares regression line was found in the form $y = Ax + B$.

It was concluded that on average, for every 4 fouls committed that the number of goals scored would increase by 1.

The total number of goals scored over the season was 15 and the fouls counted were: $1, 4, 5, 3, 0, 2, 7, 6, 10, 3$.

(a) It is known that $(\bar{x}, \bar{y})$ will be a point on the regression line. **3**

Using this fact, along with other information, find the equation of the regression line.

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Question 15 continues on page 11

(b) The analyst wants to predict the number of fouls committed by the team if 5 goals were scored. **2**

They does this by letting $y = 5$ and then solving for x.

Find **two** issues with this method. One of these methods should refer to the method behind least squares regression.

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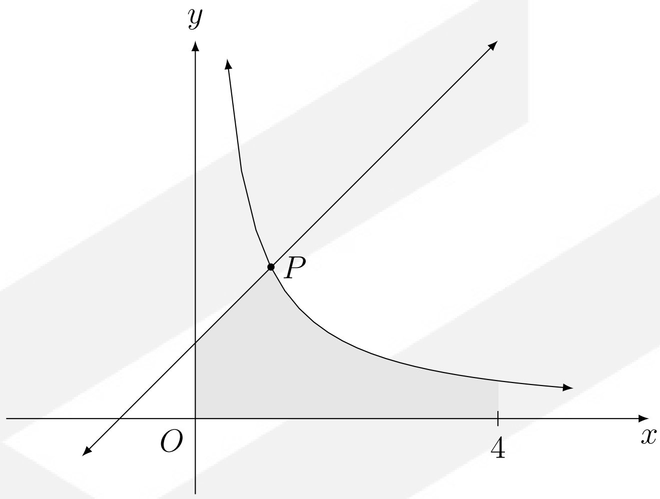
Question 16 (4 marks)

The graphs of $y = \dfrac{2}{x}$ and $y = x + 1$ intersect at the point P as shown in the diagram below. **4**



Find the area of the region bounded by the two curves, the y-axis and the line $x = 4$.

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Question 17 (4 marks)

The table below gives the future value of an annuity of $1 at the end of each period for various periods and interest rates.

Table of Future Value Interest Factors								
	Interest rate per period							
Number of Periods	0.25%	0.30%	0.35%	0.40%	0.45%	0.50%	0.55%	0.60%
53	56.5961	57.3530	58.1230	58.9063	59.7033	60.5141	61.3391	62.1785
54	57.7376	58.5250	59.3264	60.1419	60.9719	61.8167	62.6765	63.5516
55	58.8819	59.7006	60.5340	61.3825	62.2463	63.1258	64.0212	64.9329
56	60.0291	60.8797	61.7459	62.6280	63.5264	64.4414	65.3733	66.3225
57	61.1792	62.0624	62.9620	63.8786	64.8123	65.7636	66.7329	67.7204
58	62.3322	63.2485	64.1824	65.1341	66.1040	67.0924	68.0999	69.1267
59	63.4880	64.4383	65.4070	66.3946	67.4014	68.4279	69.4744	70.5415
60	64.6467	65.6316	66.6359	67.6602	68.7047	69.7700	70.8565	71.9647
61	65.8083	66.8285	67.8692	68.9308	70.0139	71.1189	72.2463	73.3965
62	66.9729	68.0290	69.1067	70.2065	71.3290	72.4745	73.6436	74.8369
63	68.1403	69.2331	70.3486	71.4874	72.6499	73.8368	75.0487	76.2859
64	69.3106	70.4408	71.5948	72.7733	73.9769	75.2060	76.4614	77.7436
65	70.4839	71.6521	72.8454	74.0644	75.3098	76.5821	77.8820	79.2101
66	71.6601	72.8670	74.1004	75.3607	76.6487	77.9650	79.3103	80.6854

A particular bank has an account with an introductory interest rate of 5.4% p.a. for the first year that drops back down to 3% p.a. after the first year. The bank compounds the interest on a monthly basis.

Alice invests 100 at the end of each month into an account earning an interest rate of 3% p.a compounded monthly for 6 years.

(a) Explain why the table cannot be used to find the value of Alice's investment after one year. **1**

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Question 17 continues on page 14

Question 17 (continued)

(b) Suppose P is the amount in the account at the end of the first year. **1**

Show that P is \$1230.15.

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(c) Using the table, along with (b), find the future value of Alice's investment. **2**

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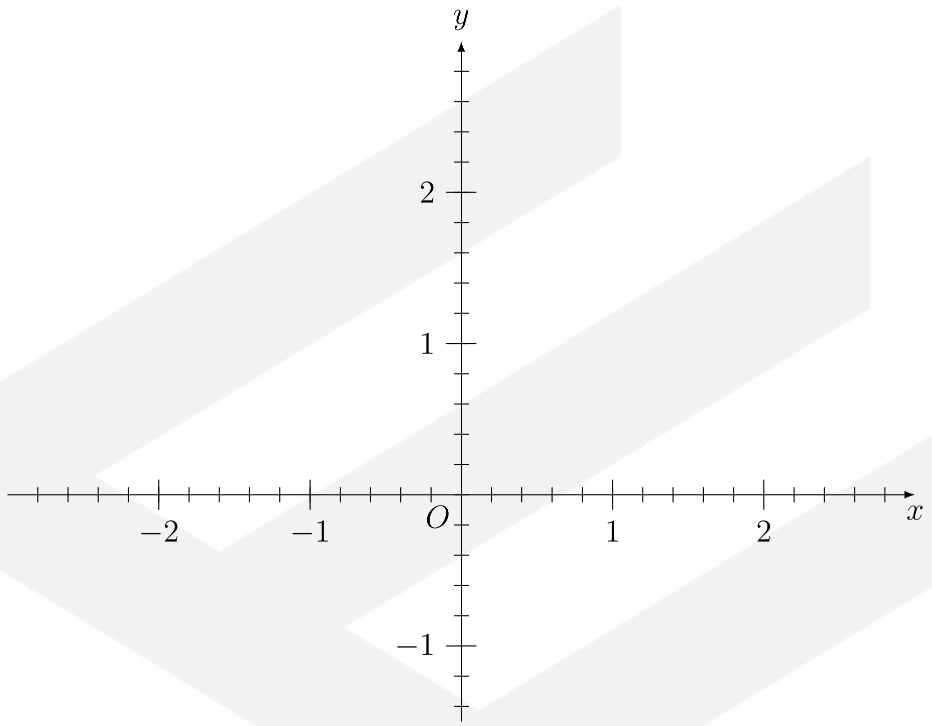
Question 18 (4 marks)

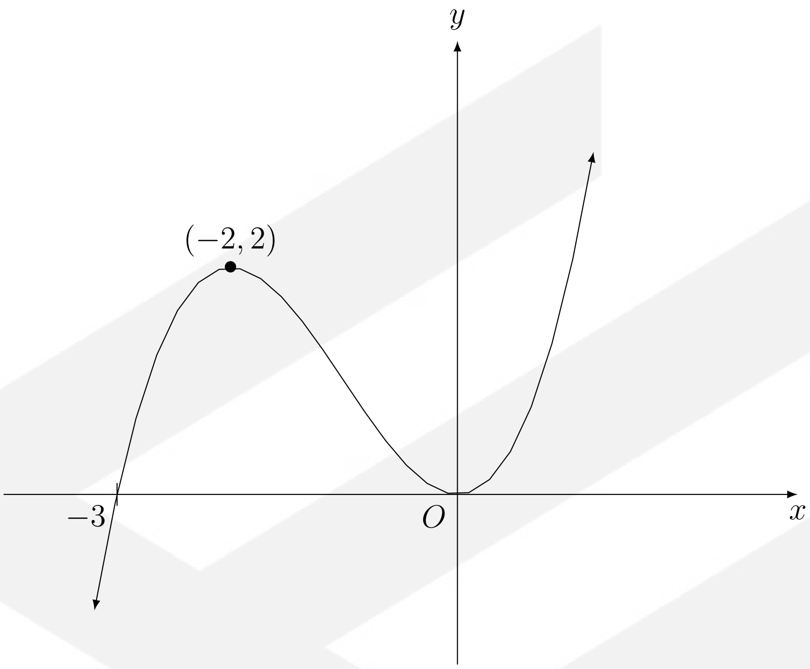
(a) Sketch the graphs of the functions $f(x) = x + 1$ and $g(x) = |2x + 1|$ on the axes below, showing x-intercepts. **2**



(b) Hence or otherwise, solve the inequality $|2x + 1| \leq x + 1$. **2**

Question 19 (3 marks)

Consider the graph of $y = f(x)$ as shown in the diagram below.



(a) Sketch the graph of $y = \frac{1}{2}f(3x)$, showing the x-intercepts and the coordinate of the turning points. **2**

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Question 19 continues on page 17

Question 19 (continued)

(b) The coordinate $P(a, b)$ is on the final graph of the transformed curve. **1**

Determine the coordinates, in terms of a and b, of the *original* point that corresponds to P under the transformation in (a).

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Question 20 (4 marks)

Some relation undergoes the following sequence of transformations: **4**

- Shifted to the right by 1 unit,
- dilated horizontally by a factor of 2 and
- contracted vertically by a factor of 3.

The resulting curve is the relation given by $(x-8)^2 + 36y^2 = 16$.
Show that the original relation was a circle and state its centre and radius.

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Question 21 (4 marks)

Find the integral $\displaystyle\int_0^{\frac{\pi}{2}} \sin x\sqrt{1+\cos x}\,dx$. **4**

Question 22 (4 marks)

At a harbour, high tide is 12 metres and low tide is 2 metres. It takes 4 hours to go from low tide to high tide. Initially at 2am, it is at low tide. Suppose the height of the tide at a harbour follows the function $x(t) = A\cos(Bt + C) + D$ for some constants A, B, C and D and t is the number of hours after 2am.

(a) Deduce that a suitable displacement time equation can be given by **2**

$$x = 7 - 5\cos\left(\frac{t\pi}{4}\right).$$

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(b) A ship needs at least at least 5 metres of water in order to safely enter the harbour. **2**

Find the latest time that the ship may leave the harbour after entering at the earliest possible time.

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Question 23 (8 marks)

The continuous random variable X has cdf for some k and $\alpha > 0$,

$$F(x) = \begin{cases} e^{-kx}\left(e^{\alpha x} - 1\right)^2 & : x \geq 0 \\ 0 & : x < 0 \end{cases}$$

(a) Show that $k = 2\alpha$. **3**

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Question 23 continues on page 21

Question 23 (continued)

(b) Suppose that the mode of the random variable X is given by m. **4**

Show that $P(X \leq m)$ does not depend on α.

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(c) Two different values are observed from this distribution. **1**

Determine the probability that exactly one value is above the mode, and the other value is below the mode.

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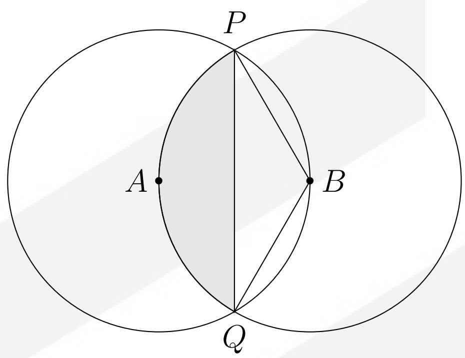
Question 24 (6 marks)

Two circles centred at A and B of radius 2cm pass through each others centres as shown in the diagram below. The circles intersect at the points P and Q and the region between the chord PQ and the arc subtended by the second circle is shaded in the diagram below.



(a) Find the size of $\angle PBQ$. **2**

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(b) Hence determine the area of the shaded region. **2**

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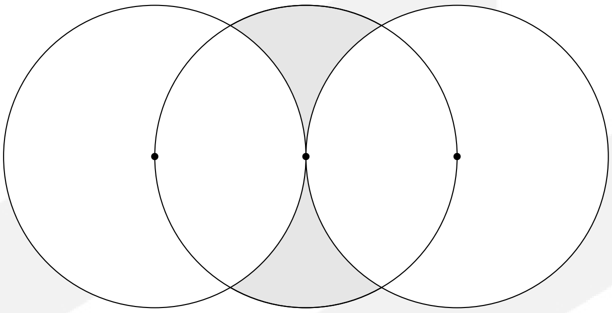
Question 24 continues on page 23

Question 24 (continued)

Now consider three circles of radius 2cm such that the two outermost circles are tangential and the middle circle passes through the centres of the outer circles. The region formed by the intersection of the outer circles with the middle circle is now shaded on the diagram below.



(c) By considering (b), or otherwise, determine the area of the new shaded region. **2**

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Question 25 (5 marks)

In a game, two fair dice are rolled and the score is the maximum of the two dice. For example if a 3 and a 5 are rolled, then the score is 5. Let the score be given by the random variable X.

(a) By considering a 6×6 grid of values, or otherwise, construct a probability distribution table for X. **2**

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(b) Hence determine the expected value of X. **1**

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Question 25 continues on page 25

(c) Hugh is offered to play this game up to two times. If he doesn't like the score in the first round, then he can choose to re-roll to try and get a better score. **2**

Hugh will re-roll if the score is less than $E(X)$. Determine his expected final score with this strategy.

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Question 26 (5 marks)

Alice inherits \$100 000 and invests it into an account earning interest at a rate of 1.2% per annum, compounded monthly. Each month, immediately after interest has been paid, Alice withdraws \$1000. **5**

The amount in the account immediately after the n^{th} withdrawal can be determined using the recurrence relation

$$A_n = A_{n-1}\,(1.001) - 1000.$$

where $n = 1, 2, 3, \ldots$ and $A_0 = 100\,000$.

In a particular month, Alice finds that she has insufficient funds to take out the usual \$1000 withdrawal and instead just takes out all of the remaining balance of $\$M$.

Determine the value of M correct to the nearest cent.

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Question 27 (5 marks)

For this question, you may use the fact that the average of n normally distributed variables with mean μ and variance σ^2 will also follow a normal distribution with mean μ and variance $\frac{\sigma^2}{n}$.

The weight of a potato is assumed to be normally distributed with mean 150 grams and standard deviation of 10 grams.

A grocery store wishes to sell bags of potatoes such that the average weight of each potato is at least 145 grams.

(a) For a bag of four potatoes, determine the probability that the average weight is no less than 145 grams. **2**

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Question 27 continues on page 28

(b) Determine the minimum number of potatoes that should be in the bag to ensure that the probability that the average weight is less than 145 grams is less than 2.5%. **3**

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Question 28 (6 marks)

Consider the function $f(x) = x^2 e^{-x}$.

(a) Find the coordinates of any stationary points and determine their nature. **2**

Question 28 continues on page 30

Question 28 (continued)

(b) Determine the behaviour of $f(x)$ as $x \to \infty$ and $x \to -\infty$. **2**

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(c) Sketch the curve, labelling the turning points and intercepts. **2**

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Question 29 (5 marks)

Alice wishes to sell an item and has an unknown reserve price X for which she will sell the item if she receives an offer that is above the reserve price (and will not sell if the offer is below the reserve price).

The reserve price will be no lower than \$500 and no higher than \$1000. It is known that the distribution of the reserve price scales linearly from 0 at 500 to most likely at 1000 such that the probability distribution of the random variable X is of the form

$$f(x) = \begin{cases} ax + b, & 500 \leq x \leq 1000 \\ 0, & \text{else} \end{cases}$$

for constants a and b.

(a) Find the value of a and b. **2**

(b) Bob knows that he can sell the item for \$1000 and so will submit an offer of x to Alice and immediately sell the item if he successfully receives it from Alice. **3**

What value of x maximises Bob's expected profit?

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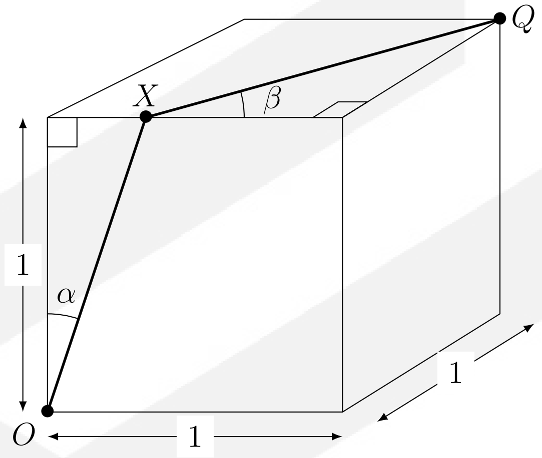
Question 30 (6 marks)

Consider an ant on the corner of a unit cube. The ant travels at an angle of α to the vertical towards the edge of the cube and then travels at an angle of β to the horizontal to the opposite corner as shown in the diagram below.



Let L be the total distance that the ant travels in this scenario.

(a) Show that the total distance is given by **2**

$$L = \sec\alpha + \sqrt{1 + (1 - \tan\alpha)^2}$$

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Question 30 continues on page 33

(b) By differentiating the expression in (a) with respect to α, find the value of α that minimises L. **3**

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Question 30 continues on page 34

Question 30 (continued)

(c) By considering the relationship between α and β, provide a geometric interpretation of the minimising value of α in terms of the net of the cube. **1**

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End of Exam