

Entwistle Mathematics



Year 12 Extension 2 Mock Exam

General Instructions:

- Reading time – 10 minutes
- Working time – 3 hours
- Write using a black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper.
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section.

Section II – 90 marks

- Attempt Questions 1-16
- Allow about 2 hours and 45 minutes for this section.

Section I

10 marks

Attempt Questions 1–10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

1 Given $z = 2 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)$, which expression is equal to $(\bar{z})^{-1}$?

(A) $\frac{1}{2} \left(\cos \frac{\pi}{3} - i \sin \frac{\pi}{3} \right)$

(B) $2 \left(\cos \frac{\pi}{3} - i \sin \frac{\pi}{3} \right)$

(C) $\frac{1}{2} \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)$

(D) $2 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)$

2 Which expression is equal to $\int \frac{1}{1 - \sin x} dx$?

(A) $\tan x + \sec x + c$

(B) $\tan x - \sec x + c$

(C) $\ln |1 - \sin x| + c$

(D) $-\frac{\ln |1 - \sin x|}{\cos x} + c$

3 Which complex number is a 6th root of i ?

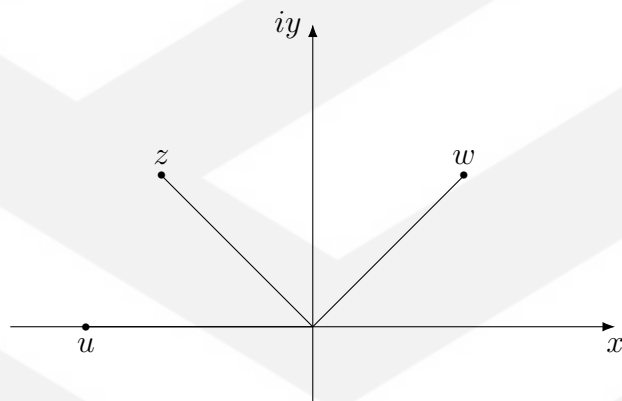
(A) $-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

(B) $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

(C) $-\sqrt{2} + \sqrt{2}i$

(D) $-\sqrt{2} - \sqrt{2}i$

4 Consider the complex numbers u, z and w as shown in the diagram below.



Which statement could be true?

(A) $u = zw$ and $u = z + w$

(B) $u = zw$ and $u = z - w$

(C) $z = uw$ and $u = z + w$

(D) $z = uw$ and $u = z - w$

5 Which integral is necessarily equal to $\int_{-a}^a f(x) dx$?

(A) $\int_0^a f(x) - f(-x) dx$

(B) $\int_0^a f(x) - f(a-x) dx$

(C) $\int_0^a f(x-a) + f(-x) dx$

(D) $\int_0^a f(x-a) + f(a-x) dx$

6 At time $t \geq 0$ seconds, a particle is at the position vector $\underline{r} = (2-t)\underline{i} - 4\sqrt{t}\underline{j} + 3\underline{k}$.

Which of the below scalars describes the particle's shortest distance from the origin?

(A) 0

(B) $\sqrt{13}$

(C) $\sqrt{23}$

(D) 6

7 Consider the statements A, B and C given by:

$A : P(x)$ is a polynomial with real coefficients

$B : \alpha$ is a complex root of $P(x)$

$C : \bar{\alpha}$ is a complex root of $P(x)$

Which of the following statements best describes the negation of the conjugate root theorem?

(A) $\neg C \vee \neg(A \vee B)$

(B) $(\neg C \vee \neg B) \longrightarrow A$

(C) $\neg C \wedge A \wedge B$

(D) $\neg(A \wedge B) \longrightarrow \neg C$

8 Let z satisfy $z^5 = |z - 1|^5$. How many distinct complex solutions exist for z ?

(A) 1

(B) 2

(C) 3

(D) 5

- 9 Suppose $z = a + ib$, where a and b are real numbers is a root of the polynomial $P(z) = a_0 + a_1iz + a_2z^2 + a_3iz^3 + a_4z^4$. It is given that a_0, a_1, a_2, a_3 and a_4 are real numbers.

Which of the following is guaranteed to also be a solution?

(A) $a - ib$

(B) $b - ia$

(C) $b + ia$

(D) $-a + ib$

- 10 Which of the following numbers will **always** be irrational for all irrational numbers $x > 0$?

(A) $(\sqrt{2})^x$

(B) $\sqrt{|\ln x|}$

(C) $x - \frac{1}{x}$

(D) $\sqrt{x+1}$

Section II

90 marks

Attempt Questions 11–16

Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available. In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks)

- (a) Consider the complex numbers $z = -9 + 3i$ and $w = 1 - 2i$. 2
- Find the modulus and the argument of the complex number $\frac{z}{w}$.
- (b) Find the area of the triangle formed by the points $(0, 0, 0)$, $(1, 2, 3)$ and $(-2, 0, 1)$ correct to two decimal places. 3
- (c) Suppose x and y are integers and consider the statement:
- “If $x^2(y + 3)$ is even, then x is even or y is odd.”
- (i) Prove that the statement is true. 2
- (ii) Write down the converse of the statement. 2
- Determine whether or not it is true or false by either providing a proof or a counter-example.
- (d) The polynomial $p(x) = x^3 + ax^2 + 6x - 3$ has a point of inflection at $x = \frac{4}{3}$. 3
- Find the three roots of $p(x)$.
- (e) Find $\int \frac{x}{\sqrt{20 + 4x - x^2}} dx$ 3

End of Question 11

Question 12 (17 marks)

- (a) Consider the statement on integers n :

$$n \geq 2 \implies \frac{1}{n} \geq \frac{1}{n+1}$$

- (i) Alice writes down the contrapositive:

2

$$\frac{1}{n} < \frac{1}{n+1} \implies n < 2$$

and claims that $n = 1$ is a counter-example since $1 < 2$ and $\frac{1}{1}$ is **not** less than $\frac{1}{1+1}$.

They use this to claim that the original statement is *false*.

Is Alice correct? If she is incorrect, then explain why.

- (ii) Bob claims that $\frac{1}{n} < \frac{1}{n+1}$ will never be true for any integer n .

2

He proves this by contradicting by multiplying both sides by $n(n+1)$ to obtain

$$\begin{aligned} \frac{n(n+1)}{n} &< \frac{n(n+1)}{n+1} \\ n+1 &< n \\ 1 &< 0 \end{aligned}$$

which is a contradiction. So Bob concludes that the original statement is *false*.

Is Bob correct? If he is incorrect, then explain why.

- (b) A particle starts at the origin from rest with acceleration given by

3

$$a = v^2 + v + 1$$

where v is the velocity of the particle.

Find x , the displacement of the particle, in terms of v .

Question 12 continues on page 9

Question 12 (continued)

(c) Consider the integral $\int_0^1 \frac{1}{x^2 + (a+2)x + 2a} dx$ for some real constant a .

(i) For what values of a is the denominator a perfect square? **2**

(ii) Suppose $a < -1$. Find the integral in terms of a . **3**

(d) (i) Sketch the region of the complex plane defined by **2**

$$\frac{\pi}{4} \leq \arg(z - 1 + i) \leq \frac{3\pi}{4}$$

(ii) Hence, or otherwise, find the minimum value of $|z + i|$ and the range of $\arg(z + i)$. **3**

End of Question 12

Question 13 (14 marks)

- (a) Prove, by contradiction or otherwise, that, if $p(x) = cx^2 + bx + a$ has no rational root, then neither does $q(x) = ax^2 + bx + c$. You may assume that a, b and c are non-zero integers for this question. **3**

- (b) A particle of unit mass is projected from the origin with speed 40m s^{-1} at an angle of 60° to the horizontal. The particle experiences forces due to gravity and air resistance which is proportional to the velocity with a constant of proportionality 2. Let the acceleration due to gravity be given by 10m s^{-2} .

The position vector at time t is given by $\underline{r}(t)$ and the velocity vector is $\underline{v}(t)$.

- (i) Prove that $\underline{v}(t) = 20e^{-2t}\underline{i} + 5((1 + 4\sqrt{3})e^{-2t} - 1)\underline{j}$ **3**

- (ii) Let the limiting horizontal displacement of the particle be given by R . **4**

If the particle has a horizontal range of $\frac{1}{2}R$ then find an expression for the maximum height of the particle.

- (c) Suppose that the complex number z lies on a circle of radius 3 centred at the complex number $2i$ in the Argand diagram.

- (i) Use the triangular inequality or otherwise to show that $|z^2 - 4| \leq 29$. **2**

- (ii) Are there any values of z for which the equality in (i) is attained? If there are, find them. If there are not, prove that no values exist. **2**

End of Question 13

Question 14 (13 marks)

- (a) Consider the statement S below: **2**

$$S : \quad (\forall x \in \mathbb{R}, \exists n \in \mathbb{N} : P(x) < n) \implies (\exists m \in \mathbb{N}, \forall y \in \mathbb{R} : P(y) < m)$$

where $P(x)$ is any polynomial with integer coefficients.

Either prove that the statement is true, or find a counterexample (and demonstrate that it is one).

- (b) How many positive integers n less than or equal 2026 satisfy the equation **3**

$$(\sin \theta + i \cos \theta)^n = \sin(n\theta) + i \cos(n\theta)$$

for all $0 \leq \theta \leq 2\pi$?

- (c) A particle is released from rest at a point P which is at some positive distance from the origin. The particle proceeds to undergo simple harmonic motion about the origin with a period T . **4**

When the particle has first passed through the origin, a second particle is released from rest at P which also then undergoes simple harmonic motion about the origin, with a larger period $2T$.

How many times do the particles intersect in the interval $0 \leq t \leq 4T$?

- (d) Consider the line ℓ_1 given by **4**

$$\underline{r}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}.$$

The line ℓ_2 intersects ℓ_1 perpendicularly at the point $(1, -3, 2)$, and also intersects the x -axis.

Determine a possible vector equation for ℓ_2 .

End of Question 14

Question 15 (14 marks)

- (a) A particle of mass M kg is launched vertically upwards from the origin with an initial speed $w \tan \alpha \text{ m s}^{-1}$ where w is the terminal velocity obtained by the particle if it had otherwise been dropped. The acceleration due to gravity is $g \text{ m s}^{-2}$ and $0 < \alpha < \pi/2$.

The particle experiences a resistive force of magnitude kMv^2 , where k is a positive constant and v is the speed of the particle after t seconds.

The maximum height reached by the particle is H metres.

(i) Show that $H = \frac{1}{k} \ln(\sec \alpha)$. 4

- (ii) When the particle lands on the ground, the speed of the particle is given by $w \sin \alpha$. **(Do NOT prove this)** 3

Show that the time taken for the particle to fall from its maximum height back to the ground is given by

$$\frac{w}{g} \ln(\sec \alpha + \tan \alpha).$$

- (b) Define for any integer $n \geq 0$, 3

$$I_n = \int \frac{1}{x^n(3x^2 + 2x - 1)} dx.$$

Find a function $f_n(x)$ (that may depend on n) and constants p and q such that I_n is of the form

$$I_n = f_n(x) + pI_{n-1} + qI_{n-2}$$

for all $n \geq 2$.

- (c) Let $H_n = \sum_{k=1}^n \frac{1}{k}$. Use mathematical induction to prove that 4

$$H_n = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \frac{1}{k}$$

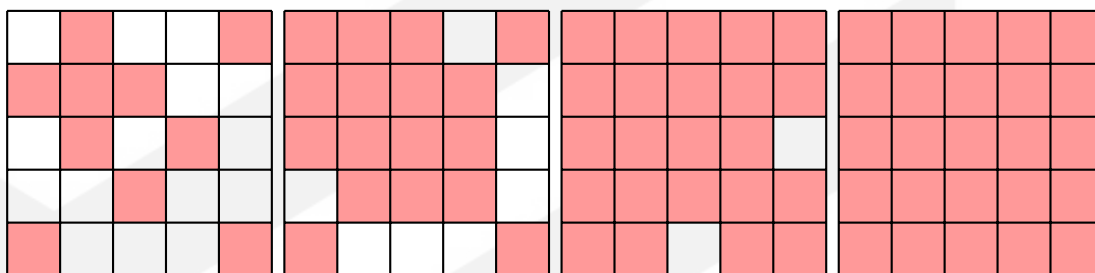
for all integers $n \geq 1$.

End of Question 15

Question 16 (17 marks)

- (a) Consider an $n \times n$ grid of n^2 cells. A mysterious disease starts at some of these cells and spreads across the grid. An *uninfected* cell will become *infected* in the next stage if at least two adjacent neighbouring cells are infected. Infected cells will remain infected forever.

An example of such an infection process is shown below which used 10 initially infected cells, and the entire grid became infected within three stages.



Let $f(n)$ be the minimum number of cells that need to be initially infected in order for the entire grid to eventually become infected.

- (i) Suppose the infection occurs at times $t = 1, 2, 3, \dots$, where $t = 0$ corresponds to the initial infection. 2

All of the infected regions form connected 'blocks'. Let $g(t)$ be the total perimeter of all of these blocks at time t . For example in the diagram above, $g(0) = 32$, $g(1) = 28$, $g(2) = 24$ and $g(3) = 20$.

The reason why $g(0) = 32$ is there is one block, , of perimeter 12 and five other blocks, , of perimeter 4.

Show that, for any initial infection configuration, $g(t)$ will always be a non-increasing function.

- (ii) Prove that $f(n) = n$. 2

- (b) Sketch the locus of the point(s) z satisfying the equation 3

$$\text{Arg}(z - 1) + \text{Arg}(z + 1) - \text{Arg}(z^2 - 1) = 2\pi.$$

justifying all working and steps used to conclude the sketch.

Question 16 continues on page 14

Question 16 (continued)

- (c) Consider a plane which intersects the positive x , y and z axes at the points $P(p, 0, 0)$, $Q(0, q, 0)$ and $R(0, 0, r)$ (none of which coincide with O), and it is known that the point $(1, 2, 3)$ lies on the plane.

- (i) Consider a variable point X on the plane. By expressing $\overrightarrow{PX}$ in terms of the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, or otherwise, deduce that **2**

$$\frac{1}{p} + \frac{2}{q} + \frac{3}{r} = 1.$$

- (ii) Show that the volume of the tetrahedron $OPQR$ is given by $V = \frac{1}{6}pqr$. **1**

- (iii) Deduce that $\left(\frac{1}{p} + \frac{2}{q}\right)^2 \geq \frac{8}{pq}$ and hence show that $V \geq 27$. **3**

- (d) Let any three digit number be denoted by xyz where x is the hundreds digit, y is the tens digit and z is the units digit.

- (i) Show that if the three digit number abc is divisible by 37, then so is bca . **2**

- (ii) Consider the five-digit number $x = 37abc$ where the five-digit numbers $37abc$, $37bac$ and $37cab$ are all multiples of 37. **2**

How many possible numbers x have this property?

End of Question 16