Entwistle Mathematics

Year 12 Extension 2 Mock Exam

General Instructions:

- Reading time – 10 minutes
- Working time – 3 hours
- Write using a black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper.
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section.

Section II – 90 marks

- Attempt Questions 1-16
- Allow about 2 hours and 45 minutes for this section.

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1−10.

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1 Consider the following statement S.

'If the dog is micro-chipped, then it has an owner'.

What is the contra-positive of the converse statement?

(A) If the dog is micro-chipped, then it has an owner

(B) If the dog doesn't have an owner, then it is not micro-chipped.

(C) If the dog is not micro-chipped, then it doesn't have an owner.

(D) If the dog has an owner, then it is micro-chipped.

2 Which of the following expressions is equal to $\int x^3 e^{-4x}\, dx$?

(A) $-\frac{3}{4}x^3e^{-4x} + \frac{3}{4}\int x^2e^{-4x}\, dx$

(B) $-\frac{3}{4}x^3e^{-4x} + \frac{3}{4}\int x^2e^{-4x}\, dx$

(C) $-\frac{1}{4}x^3e^{-4x} - \frac{3}{4}\int x^2e^{-4x}\, dx$

(D) $-\frac{1}{4}x^3e^{-4x} + \frac{3}{4}\int x^2e^{-4x}\, dx$

3 How many of the below statements are **false**?

I	$\forall x \in \mathbb{R}\backslash\{0\} : \text{If } y > \dfrac{4}{x} \text{ and } x < 1 \text{ then } y > 4$
II	$\forall x \in \mathbb{R}, \exists y \in \mathbb{C} \text{ such that } xy = 1$
III	$\exists p \in \mathbb{N} \text{ such that } p \text{ is prime } \implies p - 1 \text{ is prime}$
IV	$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Q} \text{ such that } y^2 = x$

(A) 0

(B) 1

(C) 2

(D) 3

4 Multiplying a non-zero complex number by $\dfrac{1+i}{1-i}$ results in a rotation about the origin on an Argand diagram.

Which of the following options correctly describes the rotation?

(A) Anticlockwise by $\frac{\pi}{4}$

(B) Clockwise by $\frac{\pi}{4}$

(C) Anticlockwise by $\frac{\pi}{2}$

(D) Clockwise by $\frac{\pi}{2}$

5 Suppose that ω is a non-real cube root of unity. Which of the following are false?

(A) $(1-\omega)(1-\omega^2) = 3$

(B) $1, \omega$ and ω^2 form an equilateral triangle of side length $\sqrt{3}$.

(C) $1 + \omega^k = -\omega^{2k}$ for any positive integer k.

(D) $\dfrac{a + b\omega + c\omega^2}{b + c\omega + a\omega^2} = \overline{\left(\dfrac{1}{\omega}\right)}$ for positive integers a, b and c.

6 A cubic polynomial with real coefficients has $z = 2 + 5i$ as one of its roots and has a y-intercept of -87.

It is also known that as $x \to \infty$, $P(x) \to \infty$ and that $P(x)$ is monic.

Which of the following is the coefficient of x in the polynomial?

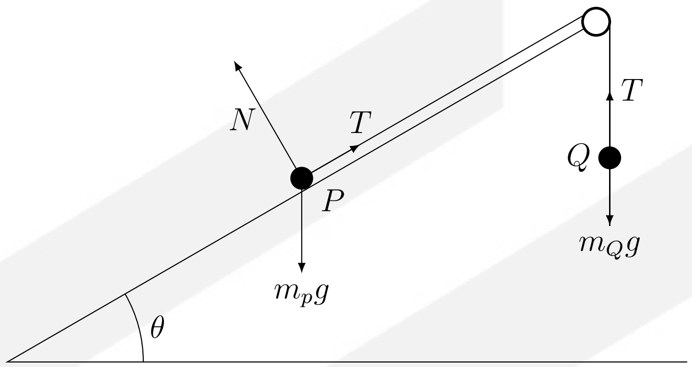
(A) -7

(B) 7

(C) 41

(D) -41

7 An object P of mass m_P sits on an inclined slope at a fixed acute angle of θ to the horizontal. It is held by a light string which extends up to the edge of the slope into a smooth pulley as shown in the diagram below. Another object Q of mass $m_Q < m_P$ is held vertically by the same light string from that pulley over the edge of the inclined slope.



The forces acting on object P are gravity with acceleration g, the tension on the string of magnitude T and a normal force of magnitude N perpendicular to the slope. Particle P is known to be sliding down the plane and hence experiences a friction force $F = \mu N$ that opposes the motion.

The net forces on each object are such that they each maintain a constant speed at any point in time.

Which of the following is the correct expression for μ?

(A) $\mu = \cot\theta - \dfrac{m_Q}{m_P}\operatorname{cosec}\theta$

(B) $\mu = \dfrac{m_Q}{m_P}\operatorname{cosec}\theta - \cot\theta$

(C) $\mu = \tan\theta - \dfrac{m_Q}{m_P}\sec\theta$

(D) $\mu = \dfrac{m_Q}{m_P}\sec\theta - \tan\theta$

8 The complex number z satisfies $|z - i| = 1$. What is the greatest distance that z can be from the point $1 - i$ on the Argand diagram?

(A) 1

(B) $\sqrt{5}$

(C) $\sqrt{5} - 1$

(D) $\sqrt{5} + 1$

9 Which of the following integrals has the largest value?

(A) $\displaystyle\int_0^{\pi} \cos^5 x \, dx$

(B) $\displaystyle\int_{-1}^{1} \frac{x^2}{1+e^x} \, dx$

(C) $\displaystyle\int_0^{1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \, dx$

(D) $\displaystyle\frac{1}{13} \int_0^{\frac{\pi}{2}} \frac{5}{1+\tan^3 x} \, dx$

10 If any person has more cats than who they are talking to – they will tell the truth, otherwise they will lie.

Alice, Bob and Charlie each have a distinct, positive integer number of cats.

- Bob says to Charlie: "You have the most cats."
- Alice says to Bob "I have 30% more cats than you."
- Alice says to Charlie "The number of cats you own is the average of how many both Bob and I own."
- Charlie says to Alice "You have no more than 3 additional cats than I do."

What is the smallest possible amount of cats that Charlie owns?

(A) 12

(B) 20

(C) 23

(D) 46

Section II

90 marks
Attempt Questions 11–-16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available. In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks)

(a) Solve the equation $x^2 + x + 1 - i = 0$. Give your answers in Cartesian form. **2**

(b) Find $\displaystyle\int_0^1 \frac{x^2}{(x+1)(x-2)^2}\,dx$ **4**

(c) Find the angle between the vectors **3**

$$\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$$
$$\underset{\sim}{b} = -\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k}$$

giving your answer to the nearest degree.

(d) A particle moves in simple harmonic motion described by the equation **3**

$$v^2 = (8x - x^2 - 7).$$

Find the magnitude of the maximum speed over the motion.

(e) Consider the two spheres $\mathcal{S}_1$ and $\mathcal{S}_2$ with respective equations given by: **3**

$$\mathcal{S}_1 : \quad (x-5)^2 + (y+6)^2 + (z-3)^2 = 49$$
$$\mathcal{S}_2 : \quad (x+3)^2 + (y-2)^2 + (z-7)^2 = 25$$

Show that the two spheres are tangential and find the point of contact.

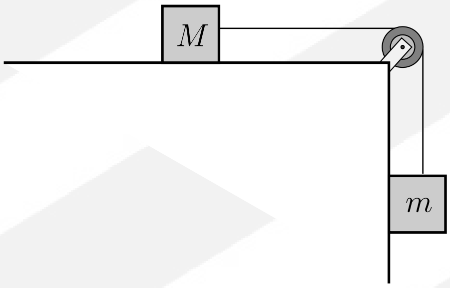
End of Question 11

Question 12 (14 marks)

(a) Find the cube roots of $8 + 8i$ in the form $re^{i\theta}$ where $-\pi < \theta \leq \pi$. **3**

(b) Two blocks of mass m and M are connected via pulley with a configuration shown in the diagram below.

The object, of mass M on the table experiences a static friction force $F = \mu N$ where N is the normal force acting on the object. There also exists a tension force T along the pulley.



(i) Show that the maximum mass for m before any sliding occurs is given by μM. **2**

(ii) Suppose that $g = 10\,\mathrm{m\,s^{-2}}$ and that $m = kM$ where $k > \mu$ such that the mass m falls. **2**

If it takes 1 second for the mass m to fall 2 metres, deduce an expression for k in terms of μ.

Question 12 continues on page 10

(c) (i) Explain why all positive numbers with k digits can be written as **1**

$$10^b A + B$$

for positive integers A and B where $B \geq 0$ and $0 \leq A \leq 10^{k-b} - 1$.

(ii) Hence, or otherwise, show that any number ending with 525 is not a perfect square. **2**

(d) The numbers x_n, for integers $n \geq 1$, are defined as **4**

$$x_1 = \sqrt{3}$$
$$x_2 = \sqrt{3 + \sqrt{3}}$$
$$x_3 = \sqrt{3 + \sqrt{3 + \sqrt{3}}}, \text{ and so on.}$$

Use mathematical induction to prove that every term in the sequence is irrational.

End of Question 12

Question 13 (14 marks)

(a) Let $f(x)$ be a continuous and differentiable function on the domain $0 \leq x \leq 1$.

(i) Show that $\displaystyle\int_0^1 f(\sqrt{x})\ dx = \int_0^1 2xf(x)\ dx.$ **1**

(ii) Hence, or otherwise, show that **3**

$$\int_0^1 (f(x))^2\ dx = \int_0^1 f(\sqrt{x})\ dx - \frac{1}{3}$$

if only if $f(x) = x$.

(b) In three-dimensional space, the lines ℓ_1 and ℓ_2 pass through the origin with direction vectors $\underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{i} + \underset{\sim}{k}$ respectively. The points A and B are given by $\lambda(1, 1, 0)$ and $\lambda(1, 0, 1)$ such that they are both a distance of $\lambda\sqrt{2}$ units from the origin.

Now consider the lines ℓ_3 and ℓ_4 that pass through the origin that makes $45°$ angles with both ℓ_1 and ℓ_2. The points P and Q lie on ℓ_3 and ℓ_4 respectively such that the distance from P and Q to the origin is 1 unit.

(i) Find the coordinates of P and Q. **3**

(ii) Show that the lines AQ and BP never intersect. **2**

(c) Suppose a, b and c are positive real numbers such that $abc = 1$.

It is known that for any real numbers x and y that

$$x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4) \quad \textbf{(Do NOT prove this)}.$$

(i) Prove that $a^5 + b^5 + ab \geq a^2b^2(a + b + c)$. **3**

(ii) Hence, or otherwise, prove the inequality **2**

$$\frac{ab}{a^5 + b^5 + ab} + \frac{bc}{b^5 + c^5 + bc} + \frac{ca}{c^5 + a^5 + ca} \leq 1$$

End of Question 13

Question 14 (16 marks)

(a) Suppose that n is a positive integer.

(i) Prove that $\left(1+\frac{1}{n}\right)^n > 2$. **1**

(ii) Hence, use mathematical induction to show that **3**

$$\left(\frac{n}{2}\right)^n > n!$$

for all integer values of $n > 5$.

(b) Consider the integral $I = \displaystyle\int_0^\infty \frac{\ln x}{x^2+nx+1}dx$.

(i) Use the substitution $u = \dfrac{1}{x}$ to show that $I = 0$ for all values of n. **2**

(ii) Hence, or otherwise, show that **3**

$$\int_0^\infty \frac{\ln x}{x^2+kx+k^2}dx = \frac{2\pi \ln k}{3\sqrt{3}k}$$

for $k > 0$.

Question 14 continues on page 13

(c) A particle of mass 2 kilograms is dropped from a height 100m above the ground.

In addition to the force due to gravity, where $g = 10\,\mathrm{m\,s^{-2}}$, the particle experiences a resistant force of kv^2 where v is the velocity of the particle and k is a constant.

(i) Show that the acceleration of the particle is given by **1**

$$a = \frac{10}{w^2}\left(w^2 - v^2\right)$$

where w is the terminal velocity of the particle if it were to never hit the ground.

(ii) Show that the velocity of the particle after t seconds is given by **3**

$$v = \frac{w\left(e^{\frac{20}{w}t} - 1\right)}{e^{\frac{20}{w}t} + 1}$$

(iii) It is known that when the particle attains 50% of the terminal velocity, it has travelled 50% of the height. **3**

Deduce in how many seconds this occurred, correct to two decimal places.

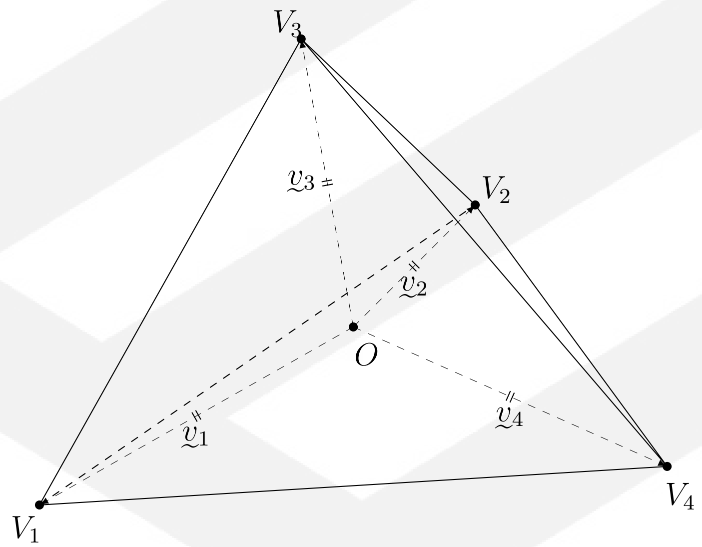
End of Question 14

Question 15 (16 marks)

(a) Recall that a regular tetrahedron is a tetrahedron with equilateral triangles of some side length s for each of its four faces.

Let V_1, V_2, V_3 and V_4 be the vertices of a regular tetrahedron circumscribed by a unit sphere.

Let the position vectors of the vertices be $\underset{\sim}{v}_1$, $\underset{\sim}{v}_2$, $\underset{\sim}{v}_3$ and $\underset{\sim}{v}_4$ respectively.



(i) Show that **3**

$$\underset{\sim}{v}_i \cdot \underset{\sim}{v}_j = \begin{cases} -\frac{1}{3} & i \neq j \\ 1 & i = j \end{cases}$$

(ii) Hence find s, the side length of the tetrahedron. **1**

(iii) Explain why there exists a cube with V_1, V_2, V_3 and V_4 as four of its vertices and find the side length of this cube. **2**

Question 15 continues on page 15

Question 15 (continued)

(b) Consider the theorem:

There is no $n \in \mathbb{Z}^+$ for which $2n-1, 5n-1$ and $13n-1$ are all simultaneously perfect squares.

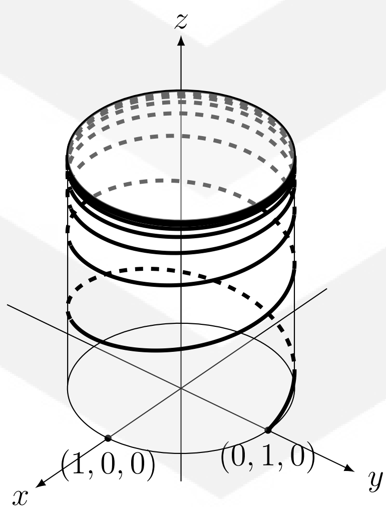
(i) Prove that if x is an integer, then x^2 is of the form $4k$ or $4k+1$ for some integer k. **2**

(ii) Suppose that $2n-1, 5n-1$ and $13n-1$ are all square numbers. **2**

Deduce that $5n-1 = 4p^2$ and $13n-1 = 4q^2$ for some integers p and q.

(iii) By obtaining a contradiction, prove the theorem. **3**

(c) Consider the spiral $\mathcal{S}$ which wraps around a cylinder of height 10 and radius 1. **3**



The spiral wraps around every 4 seconds, and the vertical distance between the start and end point on each wrap is reduced by a factor of 2 each time so that it is more densely wrapped towards the top of the cylinder.

If $P(t)$ is the position of the spiral at time t, determine some possible parametric equations for $P(t)$ for $t \geq 0$.

End of Question 15

Question 16 (15 marks)

(a) For each positive integer n, define the integrals

$$I_n = \int_0^1 \frac{2x^n}{\sqrt{1-x^2}}dx \quad \text{and} \quad J_n = \int_0^1 2x^n\sqrt{1-x^2}\,dx.$$

(i) Show that $J_n = I_n - I_{n+2}$ and hence that $I_n = \dfrac{n-1}{n}I_{n-2}$ for $n \geq 2$ **3**

(ii) Hence show that **2**

$$I_{2n} = \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \cdots \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \pi.$$

It can similarly be shown that

$$I_{2n+1} = \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} \times \cdots \times \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \times 2 \quad \textbf{(Do NOT prove this.)}$$

(iii) Prove that **2**

$$1 \leq \frac{I_{2n}}{I_{2n+1}} \leq \frac{I_{2n-1}}{I_{2n+1}}$$

(iv) Deduce that $\displaystyle\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = 1$ and hence prove that **2**

$$\left(\frac{2}{1} \times \frac{2}{3}\right) \times \left(\frac{4}{3} \times \frac{4}{5}\right) \times \left(\frac{6}{5} \times \frac{6}{7}\right) \times \cdots = \frac{\pi}{2}.$$

(v) Hence, or otherwise, prove that **3**

$$\frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \cdots \times \frac{99}{100} \geq 0.07$$

(b) Sketch the locus of the points z where $|z| = \operatorname{Arg}(z)$. **3**

End of paper