Entwistle Mathematics

Probability Exam

General Instructions:

- Reading time – 10 minutes
- Working time – 4 hours
- Write using a black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper.
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Paper has been designed to cover cumulative levels of difficulty.
  Y12 ADV students may attempt Q1–6 of Multiple Choice and then Questions 11–12.
  Y12 EXT1 students may attempt Q1-10 of Multiple Choice and then Questions 11–15.
  Y12 EXT2 students may attempt the whole paper (in theory!).

Total marks: 110

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section.

Section II – 100 marks

- Attempt Questions 1-16
- Allow about 3 hours and 45 minutes for this section.

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Consider a continuous random variable X with probability density function $f(x)$, mean μ and standard deviation σ.

The random variable Z given by $Z = \dfrac{X-\mu}{\sigma}$ has probability density function $g(z)$.

Which of the following expressions is correct for $g(z)$?

(A) $f(\mu + \sigma z)$

(B) $\sigma f(\mu + \sigma z)$

(C) $\dfrac{1}{\sqrt{2\pi}} \exp\left(-\dfrac{x^2}{2}\right)$

(D) $\dfrac{1}{\sqrt{2\pi}\sigma} \exp\left(-\dfrac{(z-\mu)^2}{2\sigma^2}\right)$

2 Which of the following is NOT a valid probability density function on the domain specified?

(A) $y = \ln x$ on $1 \le x \le e$

(B) $y = \dfrac{\sqrt{4-x^2}}{\pi}$ on $0 \le x \le 2$

(C) $y = \dfrac{x}{\ln(17)(1+x^2)}$ on $0 \le x \le 4$

(D) $y = 2023e^{-2023x}$ on $x > 0$

3 Alice takes a test to determine if she has contracted a particular disease, which is found in 1% of the population.

For people who have the disease, the test will have an accuracy of 85%. However the test will incorrectly provide a false positive 5% of the time for a person who does not carry the disease.

Alice takes the test and it returns a 'positive' result.

What is the probability, as a percentage correct to one decimal place, that Alice actually has the disease?

(A) 85.0%

(B) 14.7%

(C) 12.4%

(D) 1.2%

4 The probability that Alice wins a particular round of a game is p. The probability that Alice wins at least one out of three games is 0.488.

What is the probability that Alice wins exactly one of the three games?

(A) 0.096

(B) 0.128

(C) 0.288

(D) 0.384

5 Let A be an event with probability 50% and B be an event with probability 70%. Which of the following best describes the possibilities for $P(A|B)$?

(A) $\frac{1}{5} \leq P(A|B) \leq \frac{7}{10}$

(B) $\frac{2}{7} \leq P(A|B) \leq \frac{5}{7}$

(C) $0 \leq P(A|B) \leq \frac{1}{2}$

(D) $0 \leq P(A|B) \leq \frac{5}{7}$

6 Consider a bag of 10 apples and 15 bananas. A fruit salad is made by adding a fruit from the bag one by one without replacement until either all the apples or all the bananas are taken out.

What is the expected number of pieces of fruit in the salad (correct to the nearest whole number)?

(A) 23

(B) 21

(C) 19

(D) 17

7 Consider all distinct arrangements of the word ENTWISTLE.

How many arrangements have 'TWIST' inside the word such as in le**TWIS**e**T**n.

(A) 24

(B) 756

(C) 1512

(D) 3024

8 In how many ways can 6 couples be arranged into 3 equal groups among 3 identical circular tables where a particular couple is **not** sitting next to each other.

(A) $\dfrac{5 \times 18!}{2 \times 19} \times \left(\dfrac{5!}{6!}\right)^3$

(B) $\dfrac{5 \times 18!}{2 \times 19} \times \left(\dfrac{5!}{6!}\right)^2$

(C) $18! \times \left(\dfrac{5!}{6!}\right)^3 \times \dfrac{3}{5}$

(D) $18! \times \left(\dfrac{5!}{6!}\right)^3 \times \dfrac{3}{5} \times 3!$

9 Consider a grid that consists of 9 cells within a 3×3 square.

A team of three players, who cannot see each-other, will **randomly** select a cell to draw a cross on any of the 9 squares. The three crosses are then drawn onto the board at the same time (with the possibility of there being multiple crosses on the same cell).

The game is won if the three crosses form a row, column or diagonal.

Which of the following is the probability that the game was "won"?

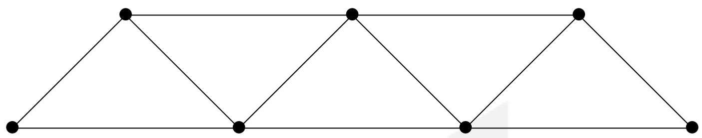
(A) $\dfrac{2}{21}$

(B) $\dfrac{34}{1701}$

(C) $\dfrac{8}{729}$

(D) $\dfrac{16}{243}$

10 Consider the shape formed by 7 vertices with edge connections between vertices as shown:



A student stands at each vertex and will travel along any connected edge with equal probability. (Students cannot remain where they are).

What is the probability that there will still be a student at each vertex?

(A) $\frac{1}{7}$

(B) $\frac{1}{49}$

(C) $\frac{1}{72}$

(D) $\frac{1}{96}$

Section II

100 marks
Attempt Questions 11–-16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available. In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (18 marks)

(a) An airline company gathers data on how quickly their planes are boarded. **3**

The time that it takes to board a plane was normally distributed with a mean of 40 minutes and a standard deviation of 15.

Approximately 26% of flights finish boarding between 25 and 35 minutes.

A particular flight has already spent 35 minutes boarding, what is the probability, correct to 2 decimal places, that they will spend at least 20 more minutes? You may *not* use any statistical tables for this question.

(b) Consider the following cumulative density function **3**

$$F(x) = \begin{cases} \dfrac{1}{A}\left(\dfrac{1}{A+1} - \dfrac{B}{Ax+1}\right), & x \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

where A and B are positive constants.

Find the possible values of A and B.

Question 11 continues on page 8

(c) Hugh wants to steal HSC questions without any copyright infringement. Each year Hugh writes a paper, and will steal one HSC question. NESA reviews the paper and will catch Hugh on his theft with probability 0.1.

(i) What is the maximum number of years that Hugh can engage in this tomfoolery if he is okay with being caught with probability 50% **1**

(ii) Suppose Hugh decides to steal from NESA 2 **times** (so writes two papers). **2**

Let X be a random variable denoting the number of times Hugh is caught.

Hugh will have to pay NESA \$40 in fines whenever he gets caught, but NESA pays a constant \$30 to their investigations department each year.

Hence NESA's 'revenue' will be given by the random variable $Y = 40X - 60$.

Find the expected value and variance of Y.

(iii) Suppose that NESA will go bankrupt if they end up with a balance of 0 or less after *each* investigation. **1**

Hugh has a secret plan to bankrupt NESA (since they don't write interesting enough probability questions) by stealing from NESA 4 **times**.

If NESA begins with \$50 in funds, what is the probability that they will go bankrupt?

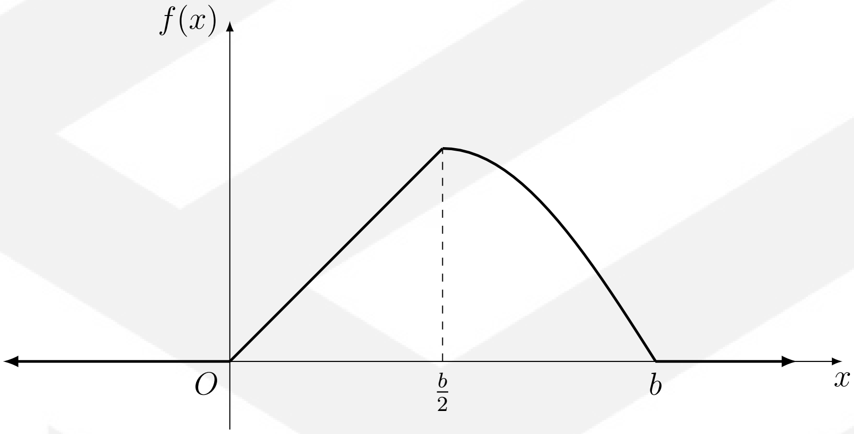
Question 11 continues on page 9

Question 11 (continued)

(d) The diagram below shows the graph of the probability density function (pdf) of the random variable X, whose expression is given by

$$f(x) = \begin{cases} cx & : 0 \le x \le \dfrac{b}{2} \\ a \sin\left(\dfrac{\pi x}{b}\right) & : \dfrac{b}{2} \le x \le b \\ 0 & : \text{otherwise} \end{cases}$$

for positive constants a, b and c.



It is known that $f(x)$ is continous at $x = \dfrac{b}{2}$.

(i) Show that **2**

$$c = \frac{8\pi}{b^2(\pi + 4)}.$$

(ii) Suppose that the median of X is m which may depend on the values of b and c. **2**

Prove that $m \ge \dfrac{b}{2}$, regardless of the values of b and c.

Question 11 continues on page 10

(e) A game involves rolling three six-sided dice one after the other. Hence we may consider the three dice as random variables D_1, D_2 and D_3.

(i) Let E be the event where D_3 is *strictly* in between D_1 and D_2. Show that the probability that the event E occurs in one roll is $\dfrac{5}{27}$. **1**

(ii) Let W be the random variable denoting the number of available 'winning' rolls for third die when the first two dice have already been observed. **1**

Use the result of (i) to show that $\mathbb{E}(W) = \dfrac{10}{9}$.

It is given that $\mathbb{E}(W^2) = \dfrac{50}{27}$. **(Do NOT prove this)**.

(iii) You will win \$100 if the event E occurs. **2**

A contract is being sold on the market that allows you to *re-roll* the third dice (if you want to). The price of this contract is x.

What is the fair value of x, correct to the nearest cent?

End of Question 11

Question 12 (15 marks)

(a) (i) Consider the sum of a sequence, S_n, given by **1**

$$S_n = a + 2ar + 3ar^2 + \cdots + nar^{n-1}$$

By considering the difference between S_n and rS_n, show that

$$S_n = a\frac{1-r^{n+1}}{(1-r)^2} + \frac{(n-1)ar^n}{1-r}.$$

(ii) Alice flips a biased coin so that the probability of the coin landing heads is p where $0 < p < 1$. **1**

Let X be a random variable denoting the number of flips Alice makes until the coin lands heads. (So for example the sequence of flips T, T, T, H results in $X = 4$).

By using (i) show that $E(X) = \dfrac{1}{p}$.

(iii) Let's deduce (ii) in a different way! **2**

Let Y be the expected number of *additional* flips required, given that the first coin was a tail. (So for example the sequence of flips T, T, T, H results in $Y = 3$).

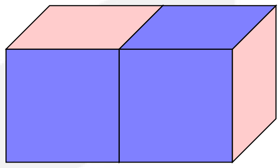
Explain why

$$X = \begin{cases} 1 & \text{with probability } p \\ 1 + Y & \text{with probability } 1 - p \end{cases}$$

and hence deduce the result of (ii).

Question 12 continues on page 12

(iv) Consider two cubes where each cube is coloured blue on five faces and red on the last face. The cubes are tossed onto the table and then moved together to form a rectangular prism, so that only 8 of the total 12 faces can be observed, with the faces joining the cubes and facing the surface of the table cannot be seen. **2**



Hugh, who hates the colour red, tosses the two cubes and counts the number of red faces that he sees. If he sees a red face, he will toss the two cubes again and repeats this process until he no longer sees any red faces.

What is the expected number of *blue faces* that Hugh will see until he stops tossing the two cubes?

Hint: It may be useful to use the previous results and choose a different value of p.

(b) Alice and Bob write down two arithmetic sequences. **2**

Suppose the n^{th} term of Alice's and Bob's sequences are given by A_n and B_n respectively.

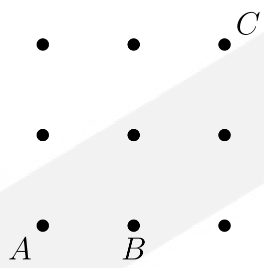
It is given that $A_n = 3n + 2$ and $B_n = 4n - 1$.

A term is chosen from Bob's sequence at random.

What is the probability that this term is also in Alice's sequence?

Question 12 continues on page 13

(c) Consider the 3×3 grid of points in the diagram below:



Each second, Alice, Bob and Charlie will move one space in a random direction (up, left, right or down) to another valid point in the grid. Two people will 'collide' if they ever land on the same point at the same time.

At time $t = 0$, Alice, Bob and Charlie initially start at the points labeled A, B and C respectively.

(i) Explain why Alice, Bob and Charlie will never intersect at the same time. **1**

(ii) It is possible for Alice and Charlie to intersect. **(Do NOT prove this.)** **3**

What is the expected number of seconds taken for them to intersect?

(d) Consider a random variable X that follows the probability density function: **3**

$$f(x) = \begin{cases} ke^{-kx}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Let $X_1, X_2, \ldots, X_n$ be independent random variables with the same distribution as X. Define the minimum and maximum as $L_n = \min\{X_1, X_2, \ldots, X_n\}$ and $U_n = \max\{X_1, X_2 \ldots, X_n\}$ respectively.

It can be shown that the distribution of L_n is equivalent to X/n where X has the same probability distribution as above. *Prove this if you feel like a challenge!*

Provide an intuitive argument as to why the random variables U_n and

$$X_1 + \frac{X_2}{2} + \frac{X_3}{3} + \cdots + \frac{X_n}{n}$$

have the *same* distribution.

End of Question 12

Question 13 (19 marks)

(a) *You may use a z-score table for this question.*

A large company is holding a Christmas party and sends out invitations to its employees. All invited employees are guaranteed to come since it is during work hours and each employee is allowed to bring up to one "plus-one" (an additional guest who was not invited). **3**

Each employee will bring a plus-one with a probability of 40%.

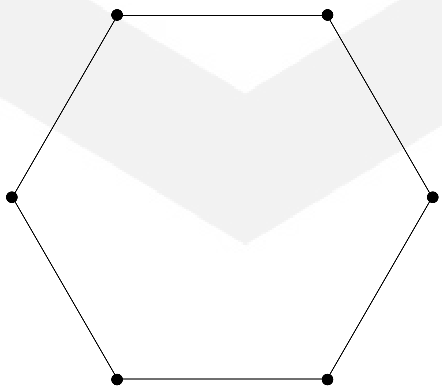
Since the venue has a strict capacity of 400 guests due to space constraints, the company deliberately invites less employees in order to accommodate for any plus-ones.

Determine the maximum number of employees that the company should invite to ensure that there is less than a 1% chance of the venue exceeding capacity.

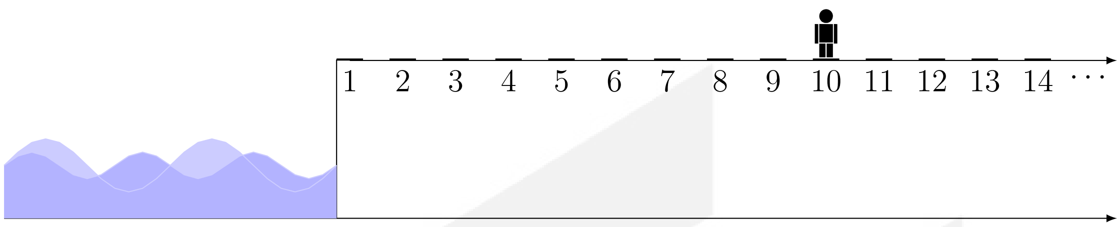
(b) Suppose that every point on the xy-plane is coloured either red, green or blue and consider the statement: **3**

There either exist two points that are precisely 1cm apart of the same colour, or there exists an equilateral triangle of some side length d that has the same colour on each vertex.

By considering the regular hexagon drawn below, or otherwise, prove this statement and specify the value of d.



(c) *Time for a walk!* A drunk sailor is ten steps away from falling into the dock as shown on the diagram below.

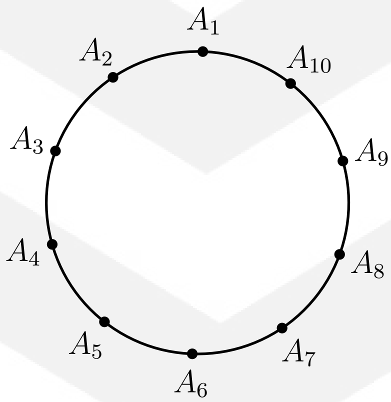


Every step he takes is either directly towards or away from the dock and he is equally likely to move in either direction.

(i) Show that the probability that he will be further from the dock after ten steps is given by $\dfrac{193}{512}$. **1**

(ii) Find the probability that he will fall in the dock on his 12^{th} step. **1**

Time to set the table! The drunk sailor now finds himself on a circle with 10 vertices that are equidistant across the circle.



Suppose that the sailor starts at vertex A_1, he will move to the left or right with equal probability on each step.

(iii) What is the probability that the sailor returns to his initial vertex after 16 steps? **2**

Time for dinner! Now five drunk sailors and their drunk wives all gather around a circular table and will all randomly sit down until all ten people are seated.

(iv) Find the probability that each sailor is sitting next to their wife. **1**

(v) Find the probability that each sailor is sitting directly opposite their wife. **2**

(d) Consider the word POSSIBILITY. All possible distinct 'words' of the eleven letters are written down and placed into a bag. Then a word is chosen from the bag.

(i) What is the probability that no two identical letters are next to eachother? (For eg: "PSOSIBILITY"). **3**

Consider all of the possible passwords that could be formed with five of the eleven letters.

(ii) For a given password, what is the probability that the no two identical letters are adjacent? **3**

End of Question 13

Question 14 (19 marks)

(a) The diagram below shows an example of a 3×3 grid. **3**

			$\cdot$
3	2	1	$= 6$
2	1	3	$= 6$
1	2	1	$\cdot$
$= 6$	$\cdot$	$\cdot$	$\cdot$

Each box of the grid is randomly filled with the number $1, 2$ or 3 where repetition of the numbers is allowed. The product across the rows, columns and main diagonals is then calculated as shown in the diagram above.

Prove that at least two of these products will be equal.

(b) Consider a slice of a family-tree with n children where each person in this 'slice' is a sibling (i.e. a brother or sister) of any other child in the 'slice'.

Assume that boys and girls are born with equal probability and that these birth events are independent.

Let B be the expected number of boy's sisters. For example in a 'slice' of 3 boys and 2 girls – the number of boy's sisters is 6.

(i) Show that $B = nE(X) - E(X^2)$ where X is a $\text{Bin}(n, p)$ random variable. **2**

(ii) Hence deduce that $B = \dfrac{n(n-1)}{4}$. **1**

Question 14 continues on page 18

(c) Let $D_{2n}(k)$ be the number of ways to sit k 'indistinguishable people' across a row of $2n$ seats with the restriction that no two people will sit next to each other.

(i) Consider the three cases: **2**

CASE 1: The first seat is taken;

CASE 2: Neither the first OR last seat is taken;

CASE 3: The first seat is not taken.

Deduce that $D_{2n}(n) = D_{2n-2}(n-1) + 1$ and hence that $D_{2n}(n) = n + 1$.

(ii) Let $x_1, x_2, x_3, \ldots, x_n$ be non-negative integers and k be a positive integer. **1**

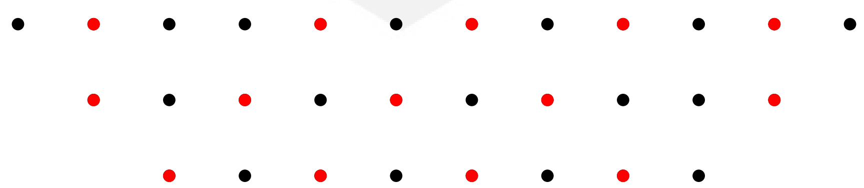
Explain why the number of solutions to the equation

$$x_1 + x_2 + \cdots + x_n = k$$

is given by $\dbinom{n+k-1}{k}$.

(iii) Using (ii) or otherwise, show that $D_{2n}(n-1) = \dfrac{n(n+1)(n+2)}{6}$. **2**

(iv) Consider the seating plan below featuring three rows of $8, 10$ and 12 seats respectively. In ideal conditions, no two adjacent seats in a row will be occupied. (One such arrangment has been shown) **2**



Find the number of ways to sit 14 *distinguishable* people into the seats with this condition.

Question 14 continues on page 19

(d) Consider a permutation of the elements $1, 2, \ldots, n$.

Let X be a random variable denoting the number of "fixed points" of a randomly chosen permutation - where a fixed point corresponds to when an element of the permutation remains in its correct position.

Let $d(n)$ correspond to the number of possible *dearrangements* of the n elements, where *none* of the elements are in their correct positions.

Note: In the 2016 Extension 2 HSC Q16(c), an explicit formula for $d(k)$ was calculated. This is just a fun fact and will **not** help with this question.

(i) Explain why the probability of the permutation containing k fixed points is given by **1**

$$p_k = \frac{\binom{n}{k} d(n-k)}{n!}$$

(ii) Show that the expected number of fixed points satisfies **2**

$$(n-1)!\,\mathbb{E}(X) = \sum_{k=0}^{n-1} \binom{n-1}{k} d(n-1-k)$$

(iii) Hence, use (ii), to find the expected number of fixed points in a randomly chosen permutation of n distinct elements. **1**

(iv) Lets do (iii) in a much better way. Let **2**

$$X_i = \begin{cases} 1 & : \text{if the } i^{\text{th}} \text{ element is a fixed point} \\ 0 & : \text{otherwise} \end{cases}$$

Use the fact that $\mathbb{E}(A + B) = \mathbb{E}(A) + \mathbb{E}(B)$ for any two random variables A and B to obtain $\mathbb{E}(X)$.

(Hint: If the answer for (iii) and (iv) do not match, then the probability that you score full marks is zero).

Question 15 (13 marks)

(a) Hugh is late for class and runs up a flight of stairs - skipping some step, so that he will either go up one step or by two steps each time (each with equal probability). Let p_N be the probability that Hugh does not skip the N^{th} step. **3**

Use mathematical induction to show that

$$p_N = \frac{2}{3} + \frac{(-1)^N}{3 \times 2^N}$$

(b) Consider the Fibonacci sequence $F_1, F_2, F_3, \ldots$ where

$$F_n = F_{n-1} + F_{n-2} \qquad \text{and} \qquad F_1 = F_2 = 1.$$

A person climbs up a staircase consisting of n stairs. In each step, they will decide to take 1 step or 2 steps.

Let A_n be the number of possible step combinations to climb the n stairs.

(i) Explain why $A_n = F_{n+1}$ and hence deduce the result **2**

$$F_{m+n} = F_{n-1}F_m + F_nF_{m+1}$$

(ii) Hence, or otherwise, deduce that F_n divides into F_{2n}. **1**

(c) Simplify the sum **2**

$$\binom{k}{k}\binom{n-k}{k} + \binom{k+1}{k}\binom{n-k-1}{k} + \cdots + \binom{n-k}{k}\binom{k}{k}$$

for $n \geq 2k$.

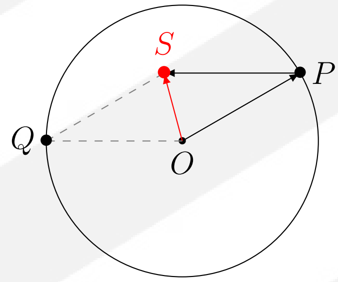
Explain your reasoning with an appropriate combinatorial argument.

Question 15 continues on page 21

(d) The drunk sailor from before has learnt how to walk in 2 dimensional space! The sailor begins at the origin of a unit circle O and will take random steps of unit length in any direction (with all directions equally likely).

We shall consider a simple 2 step journey. Suppose P and Q are randomly chosen points on the circumeference of the unit circle. Then the sailor's final position, S, is given by the vector $\overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{OQ}$.

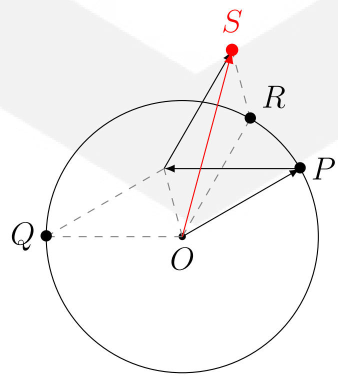
One such example is shown in the diagram below, where the sailor failed to escape the circle.



(i) Find the probability that the sailor lands *outside* the circle. **1**

That was easy enough! Let us now analyse a journey with 3 steps.

This time we randomly choose three points P, Q and R in the same manner as before. The sailors position, S, is now given as the vector $\overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{OQ} + \overrightarrow{OR}$. An example is shown in the diagram below, where this time the sailor escaped!



(ii) Explain why the probability that the sailor escapes can be modelled as **1**

$$P\left(|\overrightarrow{OZ}| > 1\right) = P\left(\left|\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix} + \begin{pmatrix}\cos\beta\\ \sin\beta\end{pmatrix}\right|^2 > 1\right)$$

where α and β are uniformly distributed angles between 0 and 2π.

Question 15 continues on page 22

Question 15 (continued)

(iii) It can be shown that **3**

$$\left|\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}\cos\alpha\\\sin\alpha\end{pmatrix}+\begin{pmatrix}\cos\beta\\\sin\beta\end{pmatrix}\right|^2=3+2\cos\alpha+2\cos\beta+2\cos(\alpha-\beta).$$

(Do NOT prove this.)

Hence find the probability that the sailor will land outside the circle.

End of Question 15

Question 16 (16 marks)

(a) Consider the function $g(N) = \frac{1}{1^N} + \frac{1}{2^N} + \frac{1}{3^N} + \frac{1}{4^N} + \frac{1}{5^N} + \cdots$ with domain $N > 1$ and consider the probability density function $p(x) = \frac{1}{x^N g(N)}$ of a random variable X for a fixed N and domain $x = 1, 2, 3, \ldots$.

(i) Let E_k be the event where X is divisible by k. Show that E_p and E_q are independent events for all prime numbers $p \neq s$. **2**

(ii) By finding the probability of some suitable event, show that **2**

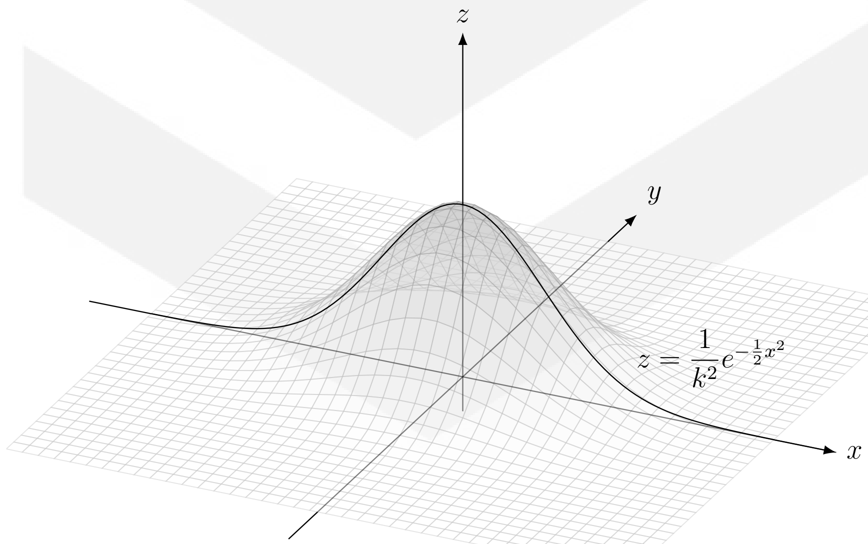
$$g(N) = \frac{1}{1-\frac{1}{2^N}} \times \frac{1}{1-\frac{1}{3^N}} \times \frac{1}{1-\frac{1}{5^N}} \times \frac{1}{1-\frac{1}{7^N}} \times \frac{1}{1-\frac{1}{11^N}} \times \cdots \times \frac{1}{1-\frac{1}{p^N}} \times \cdots$$

where the product is taken across all prime numbers, p.

(b) (i) The joint probability distribution for two independent standard normal random variables is obtained by rotating the curve in the xz-plane **3**

$$z = \frac{1}{k^2} e^{-\frac{1}{2}x^2}$$

about the z-axis to form a three-dimensional surface, as shown below.



Find the value of the positive constant k if the volume of the solid of revolution is exactly 1.

(ii) By finding an appropriate function $f(x, y)$, find the Cartesian equation for the joint probability distribution in the form $z = f(x, y)$. **1**

Question 16 continues on page 24

(c) Consider the continuous random variables X and $Y = |X - t|$ for some constant t.

Let the cumulative density function (CDF) of X and Y by given by the functions $F(x)$ and $G(y)$.

(i) Show that $G(y) = F(y - c) - F(c - y)$ for $y > 0$. **1**

(ii) The expected value of Y is given by **2**

$$E(Y) = \lim_{n \to \infty} \int_0^n y \cdot g(y)\, dy$$

where $g(y)$ is the probability density function (PDF) of Y. **(Do NOT prove this result.)**

Show that

$$E(Y) = \int_0^\infty (1 - G(y))\, dy$$

(iii) Show that the value of t that *minimises* $E(|X - t|)$ is a median of X. **2**

(iv) Hence, or otherwise, prove that for any random variable X having mean μ, variance σ^2 and median m, that **3**

$$\mu - \sigma \leq m \leq \mu - \sigma.$$

End of Question 16