Entwistle Mathematics

Year 12 Extension 2 Mock Exam

General Instructions:

- Reading time – 10 minutes
- Working time – 3 hours
- Write using a black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper.
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section.

Section II – 90 marks

- Attempt Questions 1-16
- Allow about 2 hours and 45 minutes for this section.

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

Solution:
Quick MC Answers:

Question	1	2	3	4	5	6	7	8	9	10
Answer	C	D	B	B	D	C	C	A	A	D

1 Which of the following equations does NOT model a particle in simple harmonic motion with a period of $\dfrac{\pi}{2}$ seconds?

(A) $x(t) = \cos 4t + \cos\left(4t + \dfrac{2\pi}{3}\right)$

(B) $x(t) = 2 - 3\cos^2(2t)$

(C) $x(t) = \sin^4 t - \cos^4 t$

(D) $x(t) = \sin^6 t + \cos^6 t$

2 Which of the following is purely imaginary?

(A) $\left(1 - i\sqrt{3}\right)^6$

(B) $(1 + i)^{12}$

(C) $\left(\sqrt{3} + i\right)^8$

(D) None of the above.

3 A particle with mass 5 kilograms is projected to the right with an initial velocity of $10\,\mathrm{m\,s^{-1}}$, and it experiences a resistive force of $20v^2$, where v is the velocity of the particle after t seconds. Let a be the acceleration of the particle after t seconds.

Which of the following is the velocity-displacement equation of the particle?

(A) $v = 5e^{-2x}$

(B) $v = 10e^{-4x}$

(C) $v = 20e^{-2x}$

(D) $v = 40e^{-4x}$

4 Consider the following statement:

"To watch the live-stream, you must have an internet connection."

Which of the following is equivalent, in truth, to the converse statement?

(A) If you can watch the live stream then you have an internet connection.

(B) If you can't watch the live stream then you don't have an internet connection

(C) If you don't have an internet connection then you cannot watch the live stream.

(D) None of the above statements are equivalent.

5 Which of the following integrals has the largest value?

(A) $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^3}}\,dx$

(B) $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^4}}\,dx$

(C) $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{x-x^3}}\,dx$

(D) $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{x-x^4}}\,dx$

6 Let a and b be positive numbers. Which of the following is the least value of

$$\left(a+\frac{4}{b}\right)^2+\left(b+\frac{4}{a}\right)^2$$

(A) 4

(B) 16

(C) 32

(D) 50

7 Which of the following is a negation of the statement regarding non-zero integers a and b?

$$\forall a, b \in \mathbb{Z},\ \exists r \in \mathbb{Q},\ \frac{1}{a} < r < \frac{1}{b}$$

(A) For no such a and b does there exist a rational value r such that $\dfrac{1}{a} < r < \dfrac{1}{b}$

(B) For all a and b, there exists an irrational value r such that $\dfrac{1}{a} < r < \dfrac{1}{b}$

(C) There does not exist a rational r such that for all a and b, $\dfrac{1}{a} < r < \dfrac{1}{b}$

(D) There exists some rational r such that for no values of a and b, $\dfrac{1}{a} < r < \dfrac{1}{b}$

8 Consider the set of all complex numbers z satisfying the locus equation $|2z + i| = 1$.

Which of the following best describes the locus of $w = \dfrac{1}{z}$ on the Argand diagram?

(A) A horizontal line.

(B) A vertical line.

(C) A circle.

(D) A circle with an undefined point.

9 Consider $I_n = \displaystyle\int_0^{n\pi} \frac{\sin x}{1+x}\, dx$ for positive integer values of n.

Which of the following inequalities is true?

(A) $I_1 < I_2 < I_3 < I_4$

(B) $I_1 > I_2 > I_3 > I_4$

(C) $I_2 < I_4 < I_3 < I_1$

(D) $I_2 > I_4 > I_3 > I_1$

10 Suppose there is a game with three contestants where each contestant is wearing either a red or a blue hat. Everyone can see each others hat, but they cannot see their own hat.

At the end of each hour, if anyone has figured out the colour of their hat - they will announce their hat color to everyone, win a cash prize and exit the game.

The game starts, and someone from the crowd (who can see that all of the contestants is wearing a red hat) shouts out "I can see at least one red hat!".

What will happen to the contestants if this is the only information they receive and that their logic is always perfect?

(A) Nothing will happen.

(B) All of the contestants win after one hour.

(C) All of the contestants win after two hours.

(D) All of the contestants win after three hours.

Section II

90 marks
Attempt Questions 11–-16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available. In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (16 marks)

(a) Consider the two complex numbers $z = \cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}$ and $w = 1 + i\sqrt{3}$

(i) Express zw in the form $re^{i\theta}$. **2**

(ii) Hence find the second smallest positive integer value of n for which $(zw)^n$ is purely imaginary. **3**

Solution:
(i): $z = e^{i\frac{\pi}{4}}$ and $w = 2e^{i\frac{\pi}{3}}$. So $zw = 2e^{i\left(\frac{\pi}{4}+\frac{\pi}{3}\right)} = 2e^{\frac{7\pi}{12}i}$.

(ii): $(zw)^n = 2^n e^{\frac{7n\pi}{12}i}$ where we solve $\cos\left(\dfrac{7n\pi}{12}\right) = 0$ to obtain $\dfrac{7n\pi}{12} = \dfrac{(2k+1)\pi}{2}$ which gives $n = \dfrac{6(2k+1)}{7}$ and so the second time this is an integer is when $2k+1 = 14$ which gives $n = 12$.

(b) Find the equation of a sphere whose centre is at $(2, -1, 1)$ and touches the line with parametric equation given by **3**

$$\underset{\sim}{r} = \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Solution:
The equation of the sphere is of the form $(x-2)^2 + (y+1)^2 + (z-1)^2 = r^2$ which is tangential to the line provided. Substitute the line into the sphere to obtain

$$\begin{aligned}(-4+3\lambda)^2 + (5+2\lambda)^2 + (2+\lambda)^2 &= r^2\\ 14\lambda^2 + 45 - r^2 &= 0.\end{aligned}$$

For tangency we force the discriminant of this equation to 0 which gives $r^2 = 45$. Hence the equation of the sphere is

$$(x-2)^2 + (y+1)^2 + (z-1)^2 = 45.$$

(c) Let ω be a non-real cube root of unity. Simplify **2**

$$\frac{(2+3\omega+4\omega^2)(2+\omega^2+\omega^4)}{1+3\omega+2\omega^2}$$

Solution:
Just keep applying $1+\omega+\omega^2=0$ and $\omega^3=1$ until it looks good:

$$\begin{aligned}
\frac{(2+3\omega+4\omega^2)(2+\omega^2+\omega^4)}{1+3\omega+2\omega^2} &= \frac{[3(1+\omega+\omega^2)+\omega^2-1]\,(2+\omega+\omega^2}{2(1+\omega+\omega^2)+\omega-1} \\
&= \frac{\omega^2-1}{\omega-1} \\
&= \frac{(\omega-1)(\omega+1)}{\omega-1} \\
&= \omega+1
\end{aligned}$$

Question 11 continues on page 9

(d) (i) Prove that if x^2 is divisible by 6 for some integer x, then x is also divisible by 6. **2**

(ii) Use (i) to show that $\sqrt{6}$ is irrational. **1**

(iii) Hence, or otherwise, prove that $\sqrt{2}+\sqrt{3}$ is irrational. **1**

Solution:
(i): We prove the contrapositive: "If x is not divisible by 6 then x^2 is not divisible by 6"

Then let $x = 6k + r$ where $r \in \{\pm1, \pm2, \pm3\}$ and note that $x^2 = 6(6k^2 + 2kr) + r^2$. The possible values of r^2 are $1, 4$ and 9 which are not divisible by 6 so x^2 is not divisible by 6. Hence the original statement has been proven (as it is equivalent to the contrapositive).

(ii): Suppose $\sqrt{6} = \dfrac{a}{b}$ where a, b are coprime integers, with $b \neq 0$. Then $b^2 = 6a^2$ which impliess, by (i), that b^2 and b are divisible by 6 so $b = 6c$ for some integer c. Substituting this back, we obtain $6c^2 = a^2$ which now implies that a^2 and a are divisible by 6.

Hence both a and b are divisible by 6 which contradicts their coprimality and hence $\sqrt{6}$ is irrational.

(iii): Suppose $\sqrt{2}+\sqrt{3} = r$ is a rational number. Then we can square both sides are rearrange to obtain

$$\begin{aligned} 2 + 2\sqrt{6} + 3 &= r^2 \\ \sqrt{6} &= \frac{r^2-5}{2} \end{aligned}$$

where the LHS is irrational (by (ii)) but the RHS is rational since $r^2 - 5$ and 2 are rational. This is a contradiction and hence $\sqrt{2}+\sqrt{3}$ is irrational.

Question 11 continues on page 10

(e) A student researched the technique of "strong induction" and used it to attempt to prove that $2 \times n = 0$ for every non-negative integer n. The proof is as follows: **2**

Base case. $n = 0$. Clearly $2 \times n = 0$.
Induction hypothesis. Assume that the statement holds for all integers between 0 and k (inclusive).

Inductive step. We aim to prove that, if the hypothesis is true, the statement is true for $n = k + 1$.

Consider $k+1$ and write $k+1 = i+j$, where i and j are non-negative numbers. Then, by the hypothesis, we have that $2i = 2j = 0$ so:

$$2(k+1) = 2(i+j) = 2i + 2j = 0 + 0 = 0$$

Conclusion: The base case is true. If the statement is true for the integers $0, 1, 2, \ldots, k$ then it is true for $k + 1$.

So since it is true for $n = 0$, it is true for $n = 1$. Since it is true for $n = 0, 1$ then it is true for $n = 2$. Since it is true for $n = 0, 1, 2$ then it is true for $n = 3$ and so on.

Hence, by mathematical induction, the statement is true for all non-negative integers.

Discuss the flaw with the students proof.

Solution:
Rewriting $k + 1$ as $i + j$ where i and j are non-negative does not apply for the case $k + 1 = 1$ since 1 can only be written as $1 + 0$, and so the assumption that $2 \times 1 = 0$ *cannot* be used since this is the required to prove (i.e. the student has accidentally assumed the required to prove in this step).

End of Question 11

Question 12 (15 marks)

(a) The complex number $3+i$ is a root of the polynomial $P(z) = z^4 - az^3 + bz^2 - az + 10$ where a and b are real numbers.

(i) Explain why $3 - i$ is also a zero of the polynomial $P(z)$. **1**

(ii) Find the remaining zeros of the polynomial $P(z)$ **3**

Solution:
(i) Since $P(z)$ is a polynomial with real coefficients and a complex root, then by the conjugate root theorem we can can conclude that the complex conjugate is also a root.

(ii): Since $3 - i$ and $3 + i$ are roots we can conclude that $z^2 + 6z + 10$ is a factor. By inspection we have that

$$P(z) = (z^2 + 6z + 10)(z^2 + cz + 1) \text{ for some real number } c.$$

Equating the coefficients of z^3 and z we have that $c + 6 = -a$ and $6 + 10c = -a$ where we solve simultaneously to obtain $c = 0$. Hence $P(z) = (z^2 + 6z + 10)(z^2 + 1)$ where we observe the roots $3 \pm i$ and $\pm i$.

(b) A particle has acceleration equation **4**

$$\ddot{x} = -e^{-x} - e^{-2x},$$

and is initially at the origin with velocity $v = 2\,\text{m}\,\text{s}^{-1}$.

Show that $x = \ln(2e^t - 1)$.

Solution:
First we get an expression involving v and x:

$$\begin{aligned}
v\frac{dv}{dx} &= -e^{-x}(1 + e^{-x}) \\
\int_2^v v\,dv &= -\int_0^x e^{-x}(1 + e^{-x})\,dx \\
\frac{1}{2}(v^2 - 4) &= \frac{1}{2}\left((1 + e^{-x})^2 - 4\right) \\
v^2 &= (1 + e^{-x})^2 \\
v &= 1 + e^{-x} \text{ since } v > 0
\end{aligned}$$

Then, since $v = \dfrac{dx}{dt}$, we can now obtain an equation involving x and t through:

$$\begin{aligned}\frac{dx}{dt} &= 1 + e^{-x} \\ \int_0^x \frac{1}{1+e^{-x}}\,dx &= t \\ t &= \left[\ln(1+e^x)\right]_0^x \\ &= \ln\left(\frac{1+e^x}{2}\right)\end{aligned}$$

then make x the subject to obtain the result.

(c) Find $\displaystyle\int_2^3 \frac{4}{(x-1)(x^2-1)}\,dx$. **3**

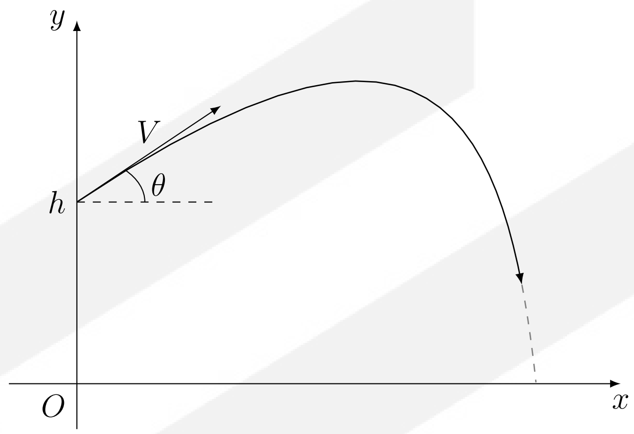
Solution:
Use partial fractions to obtain

$$\frac{4}{(x-1)(x^2-1)} = \frac{1}{x+1} - \frac{1}{x-1} + \frac{2}{(x-1)^2}$$

and then this is a routine integral to obtain $1 - \ln\left(\dfrac{3}{2}\right)$.

Question 12 continues on page 13

(d) A particle is thrown from the point O on the top of a very tall building of some height h with a fixed velocity V and variable angle θ subject to some resistance. A possible trajectory for a given h and θ is shown on the diagram below.



The equation of motion is

$$\underset{\sim}{a} = -k\underset{\sim}{v} - g\underset{\sim}{j}$$

where $\underset{\sim}{a}, \underset{\sim}{v}$ and $\underset{\sim}{r}$ are the acceleration, velocity and position vector respectively of the particle at time t. k is a positive constant corresponding to the magnitude of the resistance and g is the acceleration due to gravity.

(i) Prove that the horizontal displacement of the particle is given by **3**

$$x = \frac{V\cos\theta}{k}\left(1 - e^{-rt}\right)$$

where V and θ are the initial velocity and angle of projection respectively.

(ii) What is the closest distance to the base of the building that you could stand to ensure that you cannot be hit by the particle no matter the value of h? **1**

Solution:
(i): Start from $\ddot{x} = -k\dot{x}$ and solve the differential equation:

$$\begin{aligned} \frac{d\dot{x}}{dt} &= -k\dot{x} \\ \int_{V\cos\theta}^{\dot{x}} \frac{1}{\dot{x}}\,d\dot{x} &= -\int_0^t k\,dt \\ \ln\left|\frac{\dot{x}}{V\cos\theta}\right| &= -kt \\ \dot{x} &= V\cos\theta e^{-kt} \end{aligned}$$

Now integrate again (using $\dot{x} = \dfrac{dx}{dt}$) which routinely gets the required result.

(ii): The limiting horizontal range is $d = \dfrac{V\cos\theta}{k}$ so this would correspond to the closest distance without being hit. This is independent of h.

End of Question 12

Question 13 (15 marks)

(a) Consider the integral I_n given by

$$I_n = \int_{-\pi}^{\pi} \frac{\sin^2 \frac{nx}{2}}{\sin^2 \frac{x}{2}}\, dx$$

where n is a positive integer.

(i) Express $\cos((n+1)x) + \cos((n-1)x)$ in the form $2\cos A \cos B$. **1**

(ii) Hence, or otherwise, show that I_n forms an arithmetic progression. **3**

(iii) Hence, or otherwise, find I_{2024}. **1**

Solution:
(i): Show that $\cos((n+1)x) + \cos((n-1)x) = 2\cos x \cos(nx)$ using sum to product formulae.

(ii): Use double angle formulae in the integral to obtain:

$$I_n = \int_{-\pi}^{\pi} \frac{1-\cos nx}{1-\cos x}\, dx$$

We can solve from (i):

$$1 - \cos(n+1)x = 2(1-\cos x)\cos(nx) + 2(1-\cos(nx)) - (1-\cos(n-1)x).$$

Substitute this into the integral:

$$\begin{aligned} I_{n+1} &= \int_{-\pi}^{\pi} \frac{2(1-\cos x)\cos(nx) + 2(1-\cos(nx)) - (1-\cos(n-1)x)}{1-\cos x}\, dx \\ &= 2\int_{-\pi}^{\pi} \cos(nx)\, dx + 2I_n - I_{n-1} \\ &= 2I_n - I_{n-1} \end{aligned}$$

which gives $I_{n+1} - I_n = I_n - I_{n-1}$ and thus I_n forms an arithmetic progression.

(iii): The first term is $I_1 = 2\pi$ and the common difference is $I_1 - I_0 = 2\pi$ hence $I_n = 2\pi + (n-1)\cdot 2\pi = 2n\pi$. So $I_{2024} = 4048\pi$.

Question 13 continues on page 16

Question 13 (continued)

(b) Define the lines ℓ_1 and ℓ_2 by

$$\ell_1: \quad \underset{\sim}{r}(\lambda) = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\ell_2: \quad \underset{\sim}{r}(\mu) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

where $\underset{\sim}{v_1}$ and $\underset{\sim}{v_2}$ are the direction vectors of ℓ_1 and ℓ_2 shown above.

(i) Prove that ℓ_1 and ℓ_2 are neither parallel nor intersecting. **2**

(ii) Find a vector $\underset{\sim}{w}$ that is perpendicular to both $\underset{\sim}{v_1}$ and $\underset{\sim}{v_2}$. **1**

(iii) Consider the points P and Q on ℓ_1 and ℓ_2 respectively such that $|\overrightarrow{PQ}|$ is minimised. **2**

Using the previous parts, or otherwise, find the shortest distance between ℓ_1 and ℓ_2.

Solution:

(i): They are clearly not parallel since in order for $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ to be a scalar multiple of $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ we would be forced to conclude that $2 = 3$.

To show that they are not intersecting, we will assume for contradiction that that do. Equating components gives:

$$\lambda = 1 + \mu \qquad (1)$$
$$2 + \lambda = \mu \qquad (2)$$
$$-1 + 2\lambda = -1 + 3\mu \qquad (3)$$

Solving (1) and (3) simultanouesly obtains $\mu = 2$ and $\lambda = 3$. However in (2) we get $2 + \lambda = 5$ and $\mu = 2$ and so (2) is not satisfied. Hence the lines do not intersect.

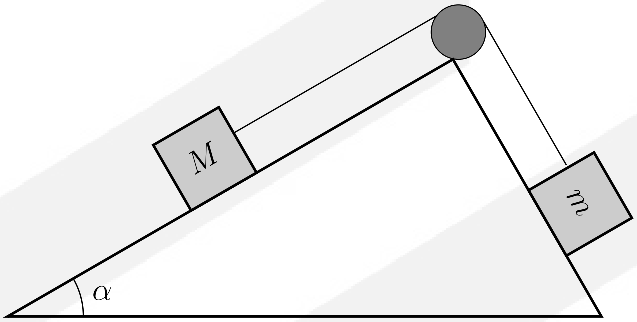
(ii): $w = (1, -1, 0)$.

(iii): The general form of $\overrightarrow{PQ}$ is $\begin{bmatrix} 1+\mu-\lambda \\ -2+\mu-\lambda \\ 3u-2\lambda \end{bmatrix}$ where the shortest distance will occur when this vector is perpendicular to both lines - i.e it will be parallel to w from (ii). Forcing this to occur we obtain $\lambda = -\dfrac{3}{2}$ and $\mu = -1$.

This gives $\overrightarrow{PQ} = \begin{bmatrix} -3/2 \\ 3/2 \\ 0 \end{bmatrix}$ (obviously a multiple of w) and the shortest distance will be the length of this vector, which is $\dfrac{3\sqrt{2}}{2}$.

Question 13 continues on page 18

(c) The diagram below shows two objects of mass M and m kilograms connected by a light inextensible string across a smooth pulley on a double-sided smooth right-angled ramp inclined at an angle of α.



(i) Show that if $\tan\alpha > \dfrac{m}{M}$ then particle A will slide down the ramp. **3**

(ii) Suppose $\tan\alpha = \dfrac{km}{M}$ where $k > 1$ such that particle A slides down the ramp. **2**

Find the value of k that maximises the acceleration.

Solution:
(i): Let a denote the acceleration of the mass M block oriented to be pointing down the plane. With this convention with tension T and N_1 and N_2 as the normal forces for the block M and m respectively, we resolve forces on both masses:

Mass M:

$$Ma = Mg\sin\alpha - T$$
$$N_1 = Mg\cos\alpha$$

Mass m:

$$ma = T - mg\cos\alpha$$
$$N_2 = mg\sin\alpha$$

Adding the first equation for both masses gives $(m+M)a = g(M\sin\alpha - m\cos\alpha)$ where $a > 0$ occurs when $g(M\sin\alpha - m\cos\alpha) > 0$ which rearranges to $\tan\alpha > \dfrac{m}{M}$ as required.

(ii): We have $a = \dfrac{g(M\sin\alpha - m\cos\alpha)}{m+M}$. Multiply the fraction by $\dfrac{k}{M}$ to obtain

$$\begin{aligned} a &= g\left(\frac{k\sin\alpha - \dfrac{km}{M}\cos\alpha}{\dfrac{km}{M}+k}\right) \\ &= g\frac{k\sin\alpha - \tan\alpha\cos\alpha}{k+\tan\alpha} \\ &= \frac{g(k-1)\sin\alpha}{k+\tan\alpha} \end{aligned}$$

Note that $\sin\alpha = \dfrac{\tan\alpha}{\sqrt{1+\tan^2\alpha}}$ and $\cos\alpha = \dfrac{1}{\sqrt{1+\tan^2\alpha}}$ which gives

$$a = \frac{gm(k-1)}{\sqrt{k^2m^2+M^2}\left(1+\dfrac{m}{M}\right)}.$$

Clearly $f(k) = \dfrac{k-1}{\sqrt{k^2m^2+M^2}}$ is an increasing function for $k>1$ and so the maximum is attained as $k \to \infty$.

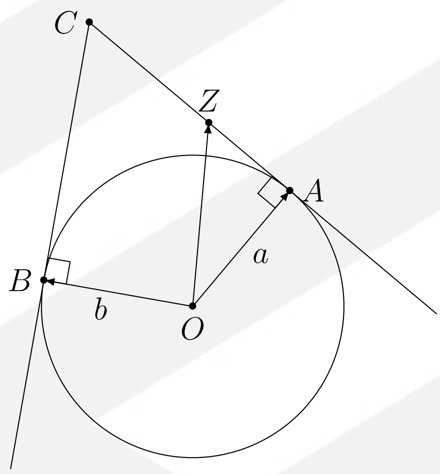
End of Question 13

Question 14 (16 marks)

(a) Let A and B be two points on the unit circle, centred at the origin, denoted by the complex numbers a and b respectively.

Let Z be a position vector on the tangent to the circle at the point A.

A tangent is also drawn at B which will intersect the other tangent at the point C as shown on the diagram below.



It can be shown that the radius of the circle is perpendicular to any tangent drawn to the circle. **(Do NOT prove this.)**

(i) Explain why $z = a(1 + ki)$ for some $k \in \mathbb{R}$. **1**

(ii) Prove that $z = 2a - a^2\bar{z}$ for all values of z on the tangent drawn to A. **3**

(iii) Show that $c = \dfrac{2ab}{a+b}$ and discuss what happens when $a = -b$ in relation to the diagram. **2**

(iv) Hence deduce the inequality $\left|\dfrac{1}{a} + \dfrac{1}{b}\right| < 2$ for any two distinct complex numbers with modulus 1 where $a + b \neq 0$. **1**

Solution:
(i): $z = \overrightarrow{OA} + \overrightarrow{AZ}$ where $\overrightarrow{AZ}$ is perpendicular to $\overrightarrow{OA}$ and hence is a scalar multiple of ai. Thus $z = a + kai = a(1 + ki)$ for some scalar $k \in \mathbb{R}$.

(ii): Consider $z + a^2\bar{z}$:

$$\begin{aligned} z + a^2\bar{z} &= a(1+ki) + a^2\bar{a}(1-ki) \\ &= a(1+ki) + a\bar{a}a(1-ki) \\ &= a(1+ki) + a(1-ki) \\ &= 2a \end{aligned}$$

Then we rearrange to obtain $z = 2a - a^2\bar{z}$.

(iii): C lies on the tangents at A and at B so we may replace a with b in (ii) to obtain the locus of z that lies on the tangent at B. Then solve the two equations simultanously to obtain C.

$$\begin{aligned} z &= 2a - a^2\bar{z} \\ z &= 2b - b^2\bar{z} \end{aligned}$$

Subtract to obtain

$$\begin{aligned} 2(a-b) - \bar{z}(a^2 - b^2) &= 0 \\ 2(a-b) - \bar{z}(a-b)(a+b) &= 0 \\ (a-b)\left(2 - \bar{z}(a+b)\right) &= 0 \end{aligned}$$

Now if $a - b \neq 0$ then we obtain $\bar{z} = \dfrac{2}{a+b}$ and so $z = \dfrac{2}{\bar{a}+\bar{b}}$. But $\bar{a} = \dfrac{1}{a}$ and $\bar{b} = \dfrac{1}{b}$ so we obtain

$$c = \frac{2}{\frac{1}{a}+\frac{1}{b}} = \frac{2}{\frac{a+b}{ab}} = \frac{2ab}{a+b}.$$

If $a \to -b$ then $|c| \to \infty$. This makes sense since as A and B become diametrically opposite in the limit, then the tangents will approach being parallel and the intersection will be further and further away (and not intersect as $a = -b$.)

(iv): The intersection is outside of the unit circle so we may apply the inequality $|c| > 1$ to obtain the required result.

Question 14 continues on page 22

(b) Consider the statement for any square with a positive side length: "The area of a square is not equal to its perimeter."

(i) How many counter-examples are there to this statement? **1**

(ii) For the rest of this question consider the statement for rectangles with positive side lengths. **2**

The statement is now: "The area of a rectangle is not equal to its perimeter."

How many counter-examples are there to this statement?

(iii) Suppose the side lengths are restricted to the positive integers. **2**

List every counter-example to the statement in (ii) with this new restriction and prove that you have found them all.

Solution:

(i): Let the side length be x. Then we can solve the equation $x^2 = 4x$ to see that $x = 4$ is the only counter-example where the area is the same as the perimeter.

(ii) Let the side length be a and b. Then we can solve the equation $ab = 2(a + b)$ to see that $a = \dfrac{2b}{b-2}$. For any value of $b > 2$ we can generate the corresponding a. So there are infinitely many counter-examples to the statement.

(iii): We have that $a(b-2) = 2b$ to form a counter-example. So either $b-2$ divides 2 (so $b = 3$ and $a = 6$) or $b - 2$ divides b. If $b - 2$ divides b then $b = k(b-2)$ for some integer k and so $b = \dfrac{2k}{k-1}$. So now $k - 1 = 1$ or $k - 1 = 2$ since $k - 1$ can't divide k since they are opposing parity. Hence $k = 2$ or $k = 3$. If $k = 2$, then $b = 4$ and $a = 4$. If $k = 3$, then $b = 3$ which we already got.

Hence the 3×6 and the 4×4 rectangles are the two only counter-examples.

Question 14 continues on page 23

(c) Consider $f(a) = a + \dfrac{1}{b(a-b)}$ which is defined for $a > b > 0$.

(i) Show that $f(a)$ is minimised when $a = b + \dfrac{1}{\sqrt{b}}$. **2**

(ii) Hence, or otherwise, deduce the minimum value of $a + \dfrac{1}{b(a-b)}$. **2**

Solution:
(i): $f'(a) = 1 - \dfrac{1}{b(a-b)^2}$ and solve $f'(a) = 0$ to obtain $a = b + \dfrac{1}{\sqrt{b}}$ and check that this minimises.

(ii): We now have that $f(a) = b + \dfrac{2}{\sqrt{b}}$ which we can optimise again with respect to b to see that this is minimal for $b = 1$. Hence the minimum value is 3.

Note that there is a nice way if we were allowed to use the degree 3 AM/GM inequality:

$$a + \frac{1}{b(a-b)} = (a-b) + b + \frac{1}{b(a-b)} \geq 3\sqrt[3]{\frac{(a-b)b}{b(a-b)}} = 3$$

End of Question 14

Question 15 (14 marks)

(a) Let $a_n = \text{Im}\left((1+2i)^n\right)$ be the imaginary part of the complex number $(1+2i)^n$ where n is a non-negative integer.

(i) Show that $a_{n+2} = 2a_{n+1} - 5a_n$ for $n \geq 0$. **2**

(ii) Prove that $a_n \neq 0$ for all $n \geq 1$. **2**

(iii) By considering the argument of $(1+2i)^n$, or otherwise, prove that $\tan^{-1}(2)$ cannot be expressed as a rational multiple of π. **2**

Solution:

(i): First we note that

$$\begin{aligned}(1+2i)^{n+2} &= (1+2i)^n(1+2i)^2\\ &= (2+2i)^n(-3+4i)\\ &= 4i(1+2i)^n - 3(1+2i)^n\\ &= 2\left[(1+2i)-1\right](1+2i)^n - 3(1+2i)^n\\ &= 2(1+2i)^{n+1} - 5(1+2i)^n\end{aligned}$$

Then we can take the imaginary part of both sides to obtain $a_{n+2} = 2a_{n+1} - 5a_n$.

(ii): Well $a_0 = 0$ and $a_1 = 2$. Since $a_1 = 2$ then we can see that a_n will always correspond to an expression of the form $2^n + 5k$ for some multiple of 5 ($k \in \mathbb{Z}$) and hence $a_n \neq 0$ since there are no multiples of 5 that are also powers of 2.

(iii): We conclude that $(1+2i)^n$ can never be real since the imaginary part is never 0. Since $(1+2i)^n = 5^{\frac{n}{2}}e^{in\tan^{-1}(2)}$ then we conclude that $n\tan^{-1}(2)$ can never be an integer multiple of π. Hence $n\tan^{-1}(2) \neq m$ for any pair of integers m and n. This implies that $\tan^{-1}(2)$ can never be in the form $\dfrac{m}{n}\pi$ and hence cannot be a rational multiple of π.

Question 15 continues on page 25

(b) Consider the sequences of real numbers $x_1, x_2, x_3, \ldots, x_n$ and $y_1, y_2, \cdots y_n$. **4**

Use mathematical induction to prove the *Cauchy-Schwarz* inequality:

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leq \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(b_1^2 + b_2^2 + \cdots + b_n^2\right)$$

for integers $n \geq 2$.

Solution:
Let $S(n)$ be the statement.

Base Case: Note that $a_1b_1 + a_2b_2$ is the dot product of the vectors (a_1, a_2) and (b_1, b_2) and so by the geometric dot product: $a_1b_1 + a_2b_2 \leq \sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}$ and then square both sides to see that $S(2)$ is true.
Assumption: Suppose that $S(k)$ is true.

Required to prove: We wish to show that $S(k) \implies S(k+1)$

$$\begin{aligned}\sqrt{LHS} &= a_1b_1 + a_2b_2 + \ldots + a_nb_n + a_{n+1}b_{n+1} \\ &= (a_1b_1 + a_2b_2 + \ldots + a_nb_n) + a_{n+1}b_{n+1} \\ &\leq \left(a_1^2 + a_2^2 + \ldots + a_n^2\right)^{1/2}\left(b_1^2 + b_2^2 + \ldots + b_n^2\right)^{1/2} \\ &+ a_{n+1}b_{n+1} \\ &\leq \left(a_1^2 + a_2^2 + \ldots + a_n^2 + a_{n+1}^2\right)^{1/2}\left(b_1^2 + b_2^2 + \ldots + b_n^2 + b_{n+1}^2\right)^{1/2} \text{ by } S(2) \\ &= \sqrt{RHS}\end{aligned}$$

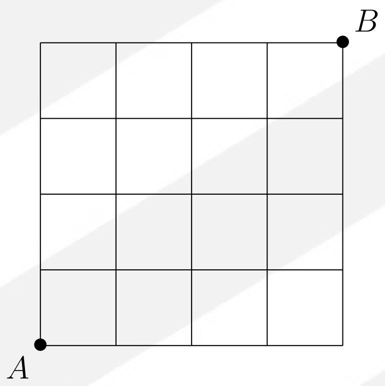
and hence $LHS \leq RHS$ as required.

Question 15 continues on page 26

(c) Consider a $n \times n$ grid with Alice and Bob beginning at opposite corners of the grid.

Alice will move towards the other corner by randomly choosing to step one unit up or to the right. Similarly Bob will move towards Alice's corner by moving one unit down or to the left.

The diagram below shows the case where $n = 4$.

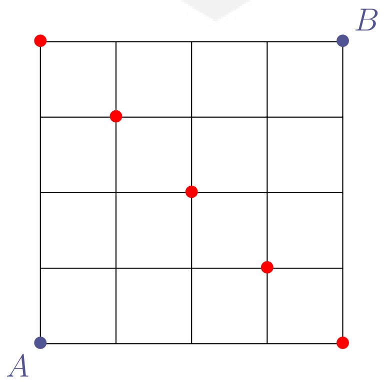


(i) Show that the probability that Alice and Bob run into eachother is given by **2**

$$p_n = \frac{1}{2^{2n}}\left(\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2\right)$$

(ii) By considering the inequality in (b) or otherwise, deduce that $p_n \geq \dfrac{1}{n+1}$. **2**

Solution:
(i): The only places where A and B can meet are along the diagonal of the board (the case for $n = 4$ is shown below as an example)



Then just do binomial theorem for each of the points with Alice and Bob both meeting there, and sum to get the result. (Remember those perms and combs questions with grids?)

$$p_n = \binom{n}{0}\frac{1}{2^n} \times \binom{n}{0}\frac{1}{2^n} + \cdots + \binom{n}{n}\frac{1}{2^n} \times \binom{n}{n}\frac{1}{2^n}$$
$$= \frac{1}{2^{2n}}\left(\binom{n}{0}^2 + \cdots + \binom{n}{n}^2\right)$$

(ii): Recall that:

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leq \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(b_1^2 + b_2^2 + \cdots + b_n^2\right)$$

So we just let $a_k = \binom{n}{k}$ and $b_k = 0$ for each $k = 0, 1, 2 \ldots, n$ to obtain

$$\left(\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}\right)^2 \leq \left(\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2\right) \times (\underbrace{1^2 + 1^2 + \cdots + 1^2}_{n+1 \text{ times}})$$

and recall that $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$. Rearrange to get

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 \geq \frac{2^{2n}}{n+1}$$

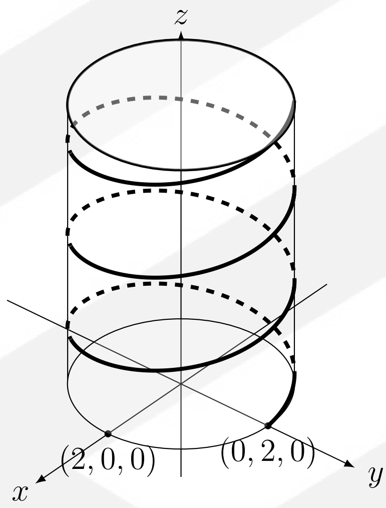
and hence $p_n \geq \dfrac{1}{n+1}$.

End of Question 15

Question 16 (14 marks)

(a) Consider a cylinder of radius 2 and height 12 such that one of its circular faces is sitting on the xy-plane with centre at the origin.

Suppose some rope is uniformly wrapped around the cylinder starting at the point $(0, 2, 0)$ and ending at the point $(0, 2, 12)$ such that it is wrapped three times around the cylinder in a anti-clockwise orientiation as shown in the diagram below.



(i) Determine the vector equation for the position of the rope $\underset{\sim}{r}(t)$ where t satisfies the domain $0 \leq t \leq 12$. **2**

(ii) An ant is at the point $(2, 0, 12)$ and travels downwards along the rope. Meanwhile a second ant is at the point $(0, 2, 0)$ travels upwards along the rope at a different speed to the first. **3**

The first ant is slow and will travel the whole length of the rope in 10 seconds, whereas the second ant will travel the whole length in 1.25 seconds.

Both ants travel at constant (but different) speeds.

Determine the time and position at which the ants will collide.

Solution:

(i): $\underset{\sim}{r}(t) = \begin{bmatrix} -2\sin\left(\dfrac{\pi t}{2}\right) \\ 2\cos\left(\dfrac{\pi t}{2}\right) \\ t \end{bmatrix}$ for $0 \leq t \leq 12$. (There are of course many other valid parameterisations).

(ii): Let the paths for the first and second ant be given by $\underset{\sim}{r_1}(t)$ and $\underset{\sim}{r_2}(t)$ respectively.

The second ant will follow the same equation in (i) but faster. So adjust such that the domain is $0 \leq t \leq 1.25$:

$$r_2(t) = \begin{bmatrix} -2\sin\left(\dfrac{24\pi t}{5}\right) \\ 2\cos\left(\dfrac{24\pi t}{5}\right) \\ \dfrac{48t}{5} \end{bmatrix} \text{ for } 0 \leq t \leq \frac{5}{4}.$$

The z component of the first ant will satisfy $z = 12 - \dfrac{12t}{10} = 12 - \dfrac{6t}{5}$ as $0 \leq t \leq 10$.

Since they are travelling opposing directions on the path - they are guaranteed to intersect (once), so it suffices to solve for when the z components are equal and then we can just use $\underset{\sim}{r_2}(t)$ to determine the intersection point.

Solving $\dfrac{48t}{5} = 12 - \dfrac{6t}{5}$ gives $t = \dfrac{10}{9}$ and then plugging this into $\underset{\sim}{r_2}(t)$ gives the intersection $\left(-\sqrt{3}, -1, \dfrac{32}{3}\right)$.

(b) It is known that the statement $A \implies B$ is false only when A is true and B is false. **3**

Use a proof by contradiction, or otherwise, to show that the statement

$$((A \implies B) \text{ and } \neg B) \implies \neg B$$

is always true.

(Note: The notation $\neg A$ stands for the negation of A).

Solution:
Suppose for contradiction that the statement is false. Then we must have that $A \implies B$ and $\neg B$ is true with $\neg B$ as false. If $\neg B$ is false then B must be true.

If $A \implies B$ and $\neg B$ is true then we must have that both the statements are true, so $A \implies B$ is true and also $\neg B$ is true. But we have already shown that $\neg B$ is false which is a contradiction.

Hence the original statement is always true.

Question 16 continues on page 30

(c) The *home prime*, $f(n)$, of a integer $n \geq 2$ is obtained by the following list of steps:

- If n is prime, then $f(n) = n$.
- If n is not prime, then concatenate the prime factors of n in ascending order (with repeats allowed) to form the new number n^*, then $f(n) = f(n^*)$.

For example $9 = 3 \times 3$ has two prime factors so $f(9) = f(33)$. Then $33 = 3 \times 11$ so $f(33) = f(311)$. Then 311 is prime, so $f(311) = 311$ and the process terminates, so 311 is the home prime of the numbers $9, 33$ and 311.

Let $f_i(n)$ be the i^{th} iteration of this process for the calculation of $f(n)$. So in the example above we have

$$f_0(9) = 9, f_1(9) = 33, f_2(9) = 311 \text{ and } f_i(9) = 311 \text{ for all } i \geq 3.$$

(i) By considering mathematical induction on the number of prime factors, (possibly duplicates) of n, prove that $f_i(n)$ is a non-decreasing sequence. **4**

(ii) It is known that 832919 is prime. **(Do NOT prove this).** **2**

Solve the equation $f(n) = 832919$ for the possible values of n. Explain why you have found all possible solutions.

Solution:

(i): **Base Case:** Suppose n only has one prime factor so $n = p$ for some prime p. Then $f(n) = n$ so $f(n) \geq n$.

Assumption: Suppose n has k prime factors (for some integer $k \geq 1$) and suppose that the concatenation K satisfies $K \geq n$.

Required to prove: Suppose N has $k+1$ prime factors.
Let p be the largest of those prime factors and assume that there are d digits for p. Then the concatenation is of the form $10^d K + p$ where K is the concatenation of k prime factors. We have two inequalities:

$10^d > p$ since p has d digits.

$K \geq \dfrac{N}{p}$ by assumption, since $\dfrac{N}{p} := n$ has k prime factors.

Hence $10^d K + p > p \times \dfrac{N}{p} + p = N + p > N$ which is enough to show that $f(N) \geq N$ since either $f(N) = N$ if N is prime, or $f(N) = f(f_1(N))$ and we have shown by induction that $f_1(N) \geq N$.

(ii): Note that the concatenation is done in ascending order. Since 8 is not prime, then we consider when 83 was the smallest prime. Then 2 and 29 are prime, but not larger than 83 but then $291 = 3 \times 97$ and $2919 = 3 \times 973$ are not primes, so 83 cannot have been the starting prime in the concatenation (since the next concatenation consists of either a smaller prime or a number that is not prime).

The next valid prime from the beginning of the sequence is 8329 (as 832 is not prime since it is even and larger than 2) but then the next possible prime after that to be concatenated with is 19 which is not larger than 8329.

The next valid prime is then 832919 (as $83291 = 13 \times 6407$ is not prime) which is just $f(n)$. (Note, we don't really need to see that 83291 is not prime, since if it was then the next concatenated numbers is 9 which isn't prime - so it wouldn't be valid anyway).

Hence there are no solutions to $f(n)$ that satisfy $n < 832919$. Clearly $n = 832919$ is a solution since $f(n) = 832919$ as 832919 is prime. Then suppose there is a solution n with $n > 832919$. This is not possible since $n \geq 832920$ and $f(n) \geq 832920 > 832919$ which would be a contradiction.

Hence the only solution is $n = 832919$.

This argument is subtle - don't be generous when giving marks to yourself here.

1 mark for proving that there are no solutions $n > 832919$ by explicitly referring to (i).

1 mark for going through the cases and demonstrating that there are no solutions $n < 832919$.

No marks awarded if a student just says $n = 832919$ is a solution (since that is obvious). Explanation needs to demonstrate that it is the *only* solution.

End of Question 16