

Entwistle Mathematics



Year 12 Advanced Mock Exam

General Instructions:

- Reading time – 10 minutes
- Working time – 3 hours
- Write using a black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper.
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section.

Section II – 90 marks

- Attempt Questions 1-29
- Allow about 2 hours and 45 minutes for this section.

Section I

10 marks

Attempt Questions 1–10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

1 Which of the following is the correct expression for $\int \frac{2}{\sqrt{7-2x}} \, dx$?

(A) $-2\sqrt{7-2x} + C$

(B) $2\sqrt{7-2x} + C$

(C) $-\sqrt{7-2x} + C$

(D) $2\sqrt{2x-7} + C$

2 Which of the following best describes the graph of $y = (0.5)^x$?

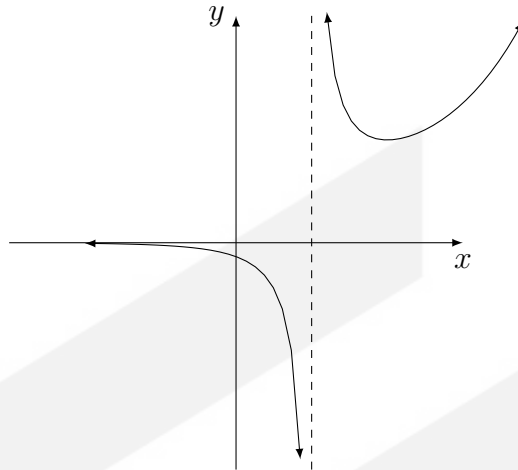
(A) A decreasing curve passing through $(0, 1)$ and approaching the x -axis as x increases.

(B) An increasing curve passing through $(0, 1)$ and approaching the x -axis as x decreases.

(C) An increasing curve passing through $(0, 1)$ and approaching the x -axis as x increases.

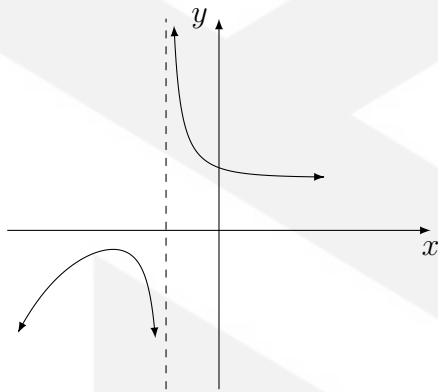
(D) A decreasing curve passing through $(0, 1)$ and approaching the y -axis as x increases.

- 3 The graph of $y = f(x)$ is shown below.

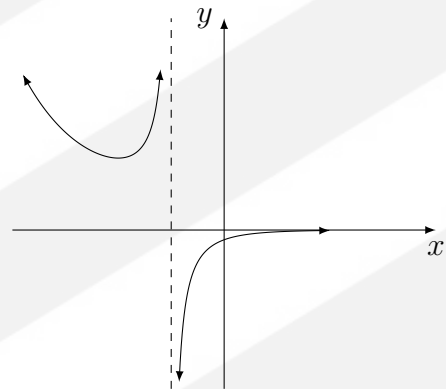


Which of the following is the graph of $y = 1 - f(-x)$?

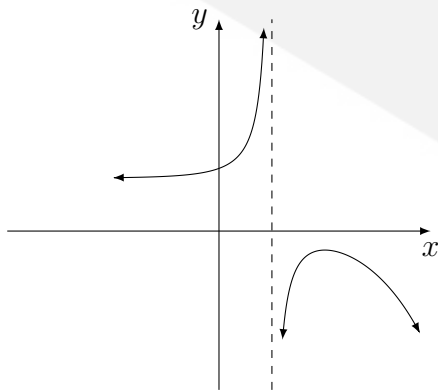
(A)



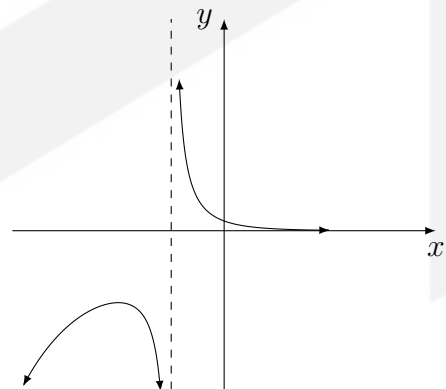
(C)



(B)



(D)



4 What is the domain of the function $y = \sqrt{2 - x^2} + \frac{1}{\sqrt{x + 3}}$?

(A) $[-\sqrt{2}, \sqrt{2}]$

(B) $(-3, \sqrt{2}]$

(C) $[-\sqrt{2}, \sqrt{2})$

(D) $(-3, \infty)$

5 A particle undergoes motion in a straight line with a displacement-time equation given by $x(t)$.

It is observed that the acceleration $\frac{d^2}{dt^2}(x(t))$ is positive and the velocity, $\frac{d}{dt}(x(t))$ is negative.

Which of the following is true about the motion?

(A) The particle is moving to the right, and speeding up.

(B) The particle is moving to the right, and slowing down.

(C) The particle is moving to the left, and slowing down.

(D) The particle is moving to the left, and speeding up.

- 6 Which of the following *cannot* be the cumulative distribution function (CDF) of some continuous random variable?

(A)
$$F(x) = \begin{cases} 0, & x \leq 1, \\ \frac{P(1 < X < x)}{P(X > 1)}, & x > 1 \end{cases}$$

(B)
$$F(x) = \frac{1}{1 + e^{-x}}, \quad -\infty < x < \infty$$

(C)
$$F(x) = \begin{cases} 0, & x < 0, \\ x + \frac{1}{4} \sin(2\pi x), & 0 \leq x \leq 1, \\ 1, & x > 1. \end{cases}$$

(D)
$$F(x) = \begin{cases} 0, & x < 0, \\ x^2, & 0 \leq x \leq 1, \\ 1, & x > 1. \end{cases}$$

- 7 The probability distribution for a discrete random variable X is shown where a, b are positive constants.

x	1	2	3	4
$P(X = x)$	0.2	0.6	a	b

Which of the following gives $P(X < E(X))$?

(A) 0.2

(B) 0.6

(C) 0.8

(D) $0.8 + a$

- 8 A fair six-sided die is rolled three times.

What is the probability that the largest number rolled is exactly 4?

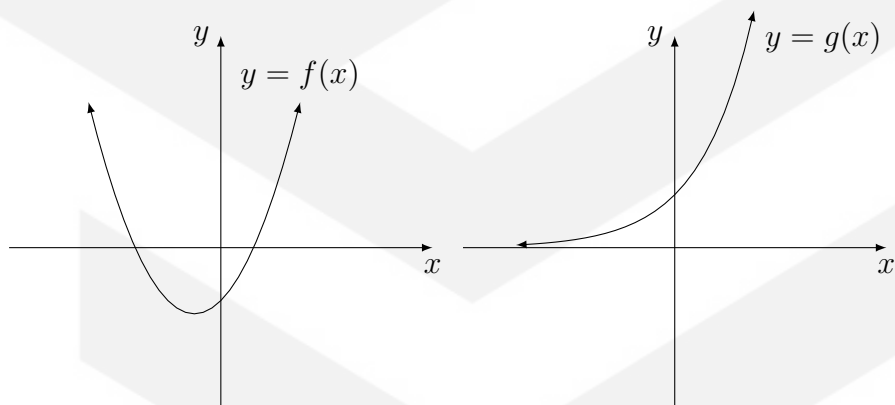
(A) $\frac{37}{216}$

(B) $\frac{19}{72}$

(C) $\frac{7}{27}$

(D) $\frac{1}{4}$

- 9 The graphs of $y = f(x)$ and $y = g(x)$ are shown below.



How many stationary points are there in the graph of $y = f(g(x))$?

(A) 0

(B) 1

(C) 2

(D) 3

- 10** Recall that the probability density function of a normally distributed random variable is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty.$$

It is known that the function $y = e^{ax^2+bx+c}$ is the probability density function of a standard normal distribution for some constants a, b and c .

Which of the following is the value of the definite integral $\int_{-\infty}^{\infty} e^{cx^2+bx+c} dx$?

(A) $\frac{1}{\sqrt{\ln(2\pi)}}$

(B) $\frac{1}{\sqrt{e}}$

(C) 1

(D) $\sqrt{\frac{e}{\ln(2\pi)}}$

Section II

90 marks
Attempt Questions 11–29
Allow about 2 hours and 45 minutes for this section

Question 11 (2 marks)

Solve the equation $2 \ln x = \ln (5x + 6)$ for the possible value(s) of x . 2

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Question 12 (3 marks)

Find the equation of the normal to $y = 5x^3 - \frac{2}{x^2} - 9$ at the point where $x = 1$. 3

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Question 13 (2 marks)

Evaluate the definite integral

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$$\int_0^1 \frac{x^2}{1+x^3} dx.$$

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Question 14 (5 marks)

A particle undergoes straight line motion with acceleration given by $a = -2 \sin\left(\frac{\pi}{3} - 2t\right)$.

The particle is initially $\frac{\sqrt{3}}{4}$ metres to the left of the origin, with an initial velocity of 0.5 m s^{-1} .

- (a) Show that the particle *does not* change direction in the first π seconds.

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Question 14 continues on page 10

Question 14 (continued)

(b) Hence, or otherwise, find the total distance travelled in the first π seconds. 2

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Question 15 (5 marks)

A school surveyed its students about whether they play a musical instrument and whether they participate in a sport. The following information is given:

- 45% of students play a musical instrument.
- 60% of students participate in a sport.
- 30% of students do both.

Let I be the event where a student plays a musical instrument and let S be the event where they participate in a sport.

(a) Show mathematically whether I and S are independent events. **2**

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(b) One student is selected at random and *doesn't* play an instrument. What is the probability that they participate in sport? **1**

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Question 15 continues on page 12

Question 15 (continued)

- (c) Two students are selected independently at random and at least one of them doesn't play an instrument. What is the probability that they *both* participate in sport? **2**

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Question 16 (6 marks)

Consider the function $f(x) = x^3e^{-x}$.

- (a) Find the stationary points and determine their nature. **3**

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Question 16 continues on page 13

Question 16 (continued)

(b) Find any points of inflection that were not found in (a). 2

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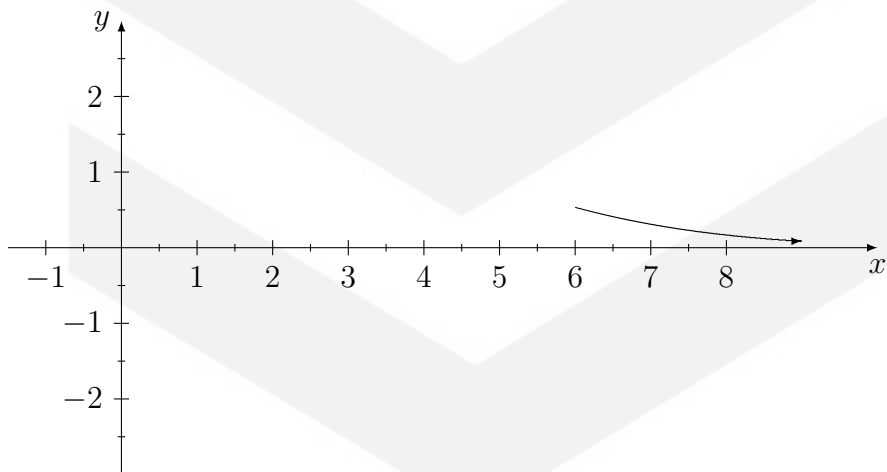
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(c) A partially completed graph of $y = f(x)$ is shown. 1



Use your answer from (a) to complete the graph.

Question 17 (5 marks)

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A vertical flagpole BD , lies on flat level ground, where B is on the ground and D is the top of the flagpole. It is supported by two cables, AD and CD . The cable AD is inclined at 30° to the horizontal and is tied in the east direction from the foot of the pole. The second cable is tied at a point, C which is south of B and on a bearing of 210° from A .

Let the angle of elevation from the base C to the top of the tower be given by θ .

Show that $\theta = \tan^{-1}\left(\frac{1}{3}\right)$.

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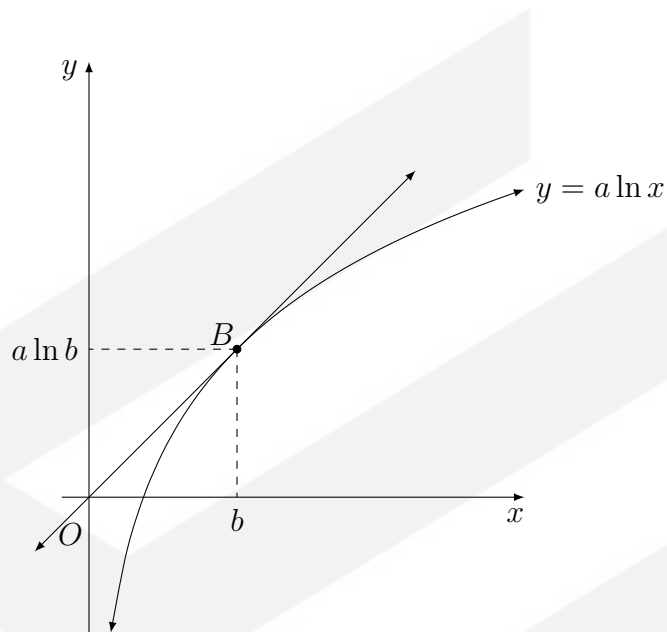
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Question 18 (4 marks)

Consider the graph of $y = a \ln x$ for $a > 0$ and a point B on the curve with coordinates $(b, a \ln b)$.

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Determine the values of a and b so that the tangent drawn at B has a gradient of 1 and passes through the origin.

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Question 19 (6 marks)

Alice borrows \$20,000 and is charged interest at a rate of 6% per annum, compounded monthly. She repays \$800 at the end of every quarter. Let A_n be the balance owing immediately after the n^{th} repayment, so that $A_n = aA_{n-1} - 800$ for some constant a .

- (a) Show that $a = 1.015$, correct to 3 decimal places. **1**

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Proceed with the rest of the question by using $a = 1.015$.

- (b) How many quarters does it take for Alice to *fully* repay the loan? **3**

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Question 19 continues on page 17

Question 19 (continued)

- (c) How much should her quarterly repayments be so that she can repay the loan by the end of the six years? **2**

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Question 20 (4 marks)

Suppose a relation undergoes a translation to the right by 3 units, and then a vertical dilation by a factor of 2. The resulting curve is the relation $4x^2 - 24x + y^2 + 4y + 28 = 0$. **4**

Show that the original relation was a circle, stating its centre and radius.

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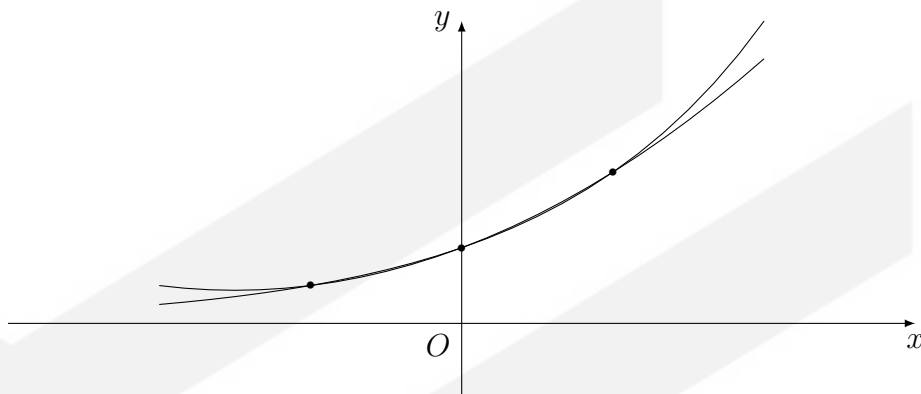
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Question 21 (4 marks)

Bob wishes to approximate the graph of $y = 2^x$ by constructing a quadratic $y = ax^2 + bx + c$ over the interval $x \in [-1, 1]$. He draws the graphs so that they both intersect at $x = -1$, $x = 0$, and $x = 1$ as shown in the diagram below.



- (a) Find the coefficients of the quadratic.

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- (b) By considering the area under the curve on $-1 \leq x \leq 1$, find a rational approximation for $\ln(2)$.

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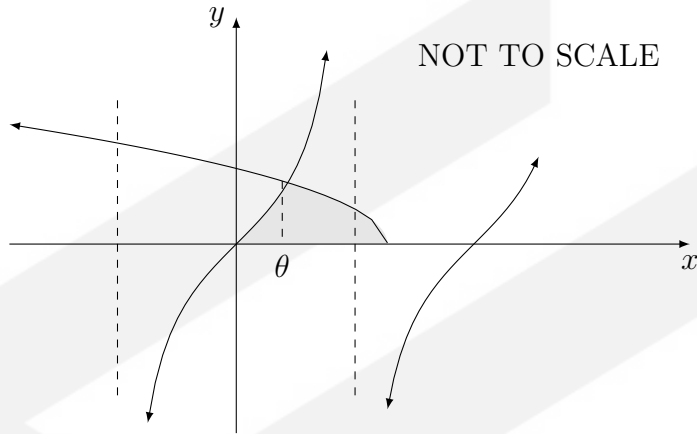
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Question 22 (4 marks)

Consider the region bounded between the graphs of $y = \tan(2x)$, $y = \sqrt{1-x}$ and the x -axis as shown in the diagram below. It is known that the graphs intersect at $x = \theta$ where $\theta > 0$.

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Show that the shaded region has area given by

$$A = \frac{1}{4} \ln(2 - \theta) + \frac{2}{3} \tan^3(2\theta).$$

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Question 23 (6 marks)

The amount of hot water (in mL) dispensed by a vending machine is assumed to be normally distributed with mean 250 mL and an unknown standard deviation σ mL.

It was noticed that 75% of the time, the machine dispensed more than 246 mL of water and 16% of the time, the machine dispensed more than 256 mL of water.

- (a) By using an appropriate empirical rule, deduce that $\sigma \approx 6$. **1**

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- (b) Find the probability that the amount dispensed is between 238 mL and 254 mL. **2**

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Question 23 continues on page 21

Question 23 (continued)

- (c) A cup of tea is made by taking *half* of whatever water is dispensed by the machine. **2**
The amount of water in the cup is still assumed to be normally distributed.

State the mean and variance of X .

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- (d) Find two values a and b for which **1**

$$P(a \leq X \leq b) \approx 49.85\%$$

where X is the amount of hot water in the cup.

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Question 24 (7 marks)

A piece of wire is L cm long. Some of the wire is used to make a circle of radius r cm. The remainder of the wire is used to make a right-angled isosceles triangle.

- (a) Show the combined area of the circle and the triangle is given by **2**

$$A(r) = \pi r^2 + \frac{1}{4(3 + 2\sqrt{2})}(L - 2\pi r)^2$$

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- (b) Show that the value of r that minimises $A(r)$ is given by **3**

$$r = \frac{L}{2(3 + \pi + 2\sqrt{2})}.$$

Explain why you know it is a minimum.

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Question 24 continues on page 23

Question 24 (continued)

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- (c) For the value of r in (b), show that the perimeter of the circle will be approximately 53.9% of the triangle's perimeter, regardless of the original length L .2

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Question 25 (6 marks)

Consider the graphs of $y = kx$ and $y = ax(1 - x)$ where $a > 0$.

- (a) Explain why $k < a$ is a necessary condition for the curves to have two points of intersection with non-negative x coordinates. **1**

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- (b) Suppose $k < a$. Show that the area bounded by the two curves is given by **3**

$$A = \frac{1}{6}a \left(1 - \frac{k}{a}\right)^3$$

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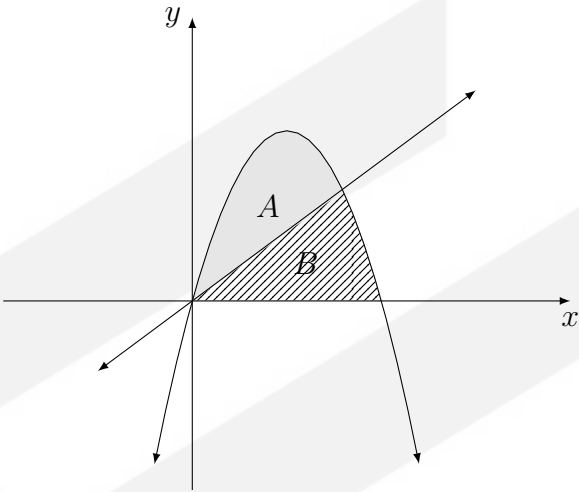
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Question 25 continues on page 25

Question 25 (continued)

- (c) Suppose k is chosen such that the line $y = kx$ slices the region under $y = ax(1 - x)$ and the x -axis into two equal regions A and B as shown in the diagram below. **2**



Show that $k = \left(1 - \frac{1}{\sqrt[3]{2}}\right)a$.

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Question 26 (6 marks)

- (a) Sketch the graph of $y = 1 - \frac{1}{2x}$, showing equations of asymptotes and intersection with the coordinate axes. **2**

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- (b) Let X be a non-negative continuous random variable with cumulative distribution function given by $F(x)$. Let m be a median of X . **2**

Show that $P(X > m | X < 2m) = 1 - \frac{1}{2F(2m)}$.

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Question 26 continues on page 27

Question 26 (continued)

- (c) By considering the graph in (a), show that

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$$P(X > m | X < 2m) \leq \frac{1}{2}.$$

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- (d) Provide an intuitive explanation of the result in (c).

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Question 27 (4 marks)

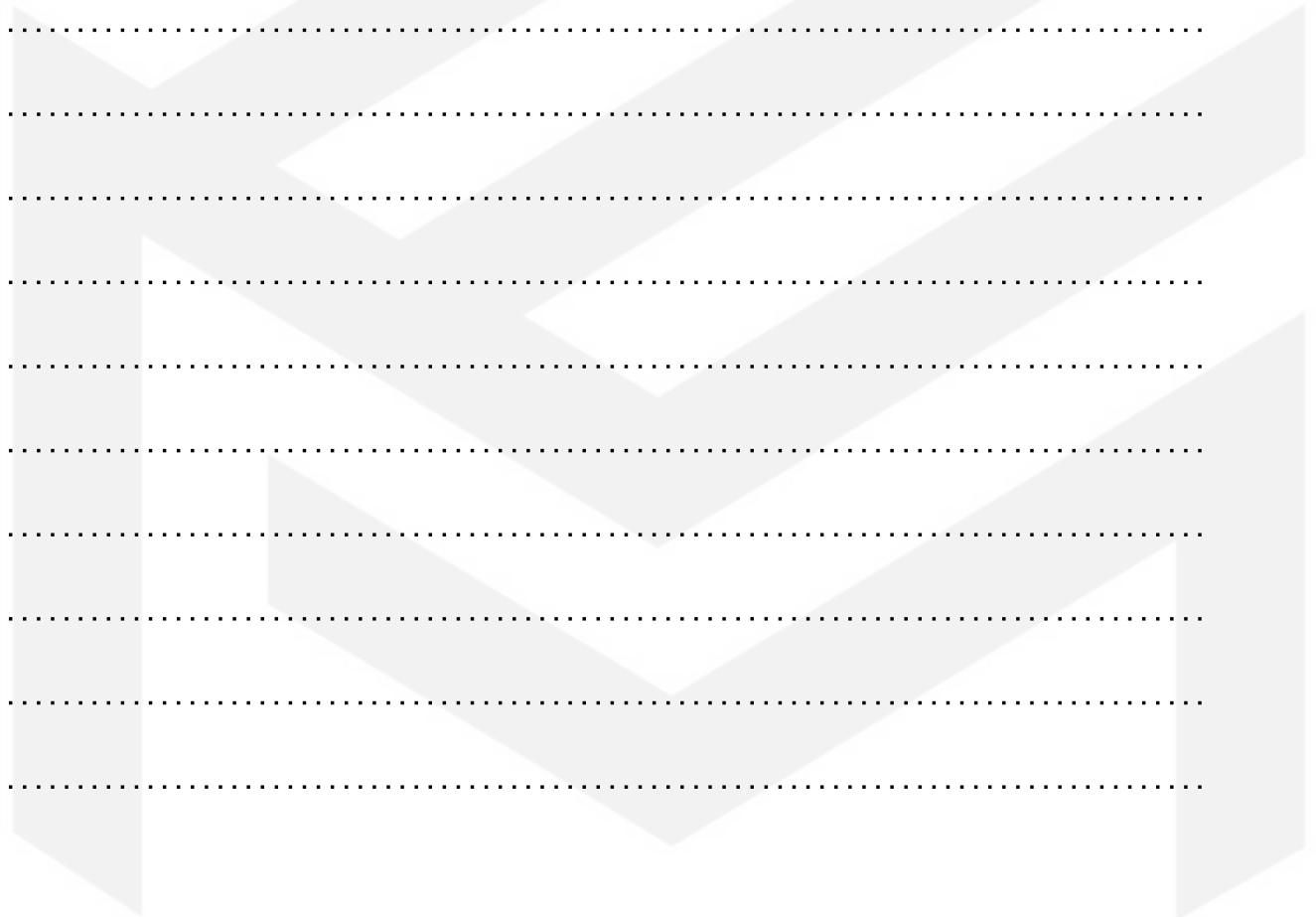
Consider the function $f(x)$ given by

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$$f(x) = \begin{cases} k + \frac{x}{k} & : -\frac{1}{k} \leq x \leq k \\ 0 & : \text{otherwise} \end{cases}$$

where k is a non-zero constant.

Show that $f(x)$ cannot be a valid probability density function.



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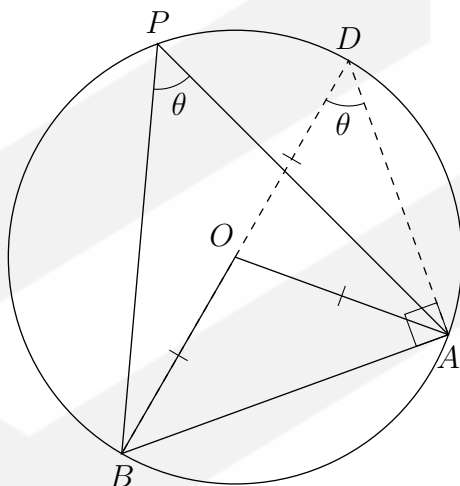
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Question 28 (2 marks)

Consider a circle with centre O and two points A and B as well as an arbitrary point P on the circumference where the arc AB subtends an angle θ . Let D be the point such that BD is a diameter.

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It is known that equal arcs subtend equal angles to any point on the circumference in the same segment. So it can be assumed that $\angle BDA = \theta$. (**Do not prove this**).

Deduce that $\angle AOB = 2\theta$.

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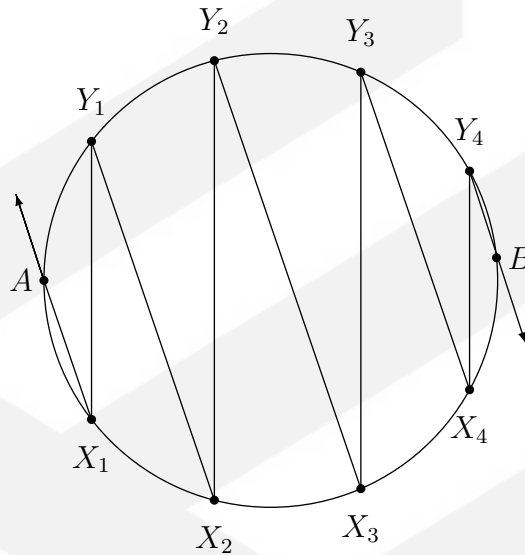
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Question 29 (9 marks)

A company commissions an artist to create a new logo by constructing a sequence of parallel lines of negative gradient m on a unit circle, starting from the point $A(-1, 0)$ and vertical lines to create a zig-zag pattern on the unit circle $x^2 + y^2 = 1$, until the zig-zag pattern goes through the last point B as shown in the diagram below.



The above diagram shows a zig-zag pattern that intersects the circle 10 times.

- (a) By considering a pair of parallel lines, explain why $\angle AX_1Y_1 = \angle X_1Y_1X_2$.

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Question 29 continues on page 31

Question 29 (continued)

- (b) By considering a different pair of parallel line, explain why $\angle X_1Y_1X_2 = \angle Y_1X_2Y_2$. 1

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- (c) Explain why the arc lengths AX_1 and AY_1 are equal. 1

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- (d) By using the fact that equal angles at the circumference of a circle subtend equal arcs, along with (a), (b) and (c), explain why the arc lengths 2

$$AX_1 = X_1X_2 = \cdots X_4B = AY_1 = Y_1Y_2 = \cdots = Y_3Y_4.$$

That is, deduce that all of the arc lengths between the lines on the artwork, except possibly Y_4B , are all equal.

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Question 29 continues on page 32

Question 29 (continued)

(e) Let $m = \tan \alpha$ where $\alpha \in (\frac{\pi}{2}, \pi)$.

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Show that the artwork will have 10 intersections if $\frac{3\pi}{5} \leq \alpha < \frac{11\pi}{18}$.

You may use *any* results that have been given or shown throughout this exam. *Good luck!*

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End of Exam