Entwistle Mathematics

Year 12 Extension 2 Mock Exam

General Instructions:

- Reading time – 10 minutes
- Working time – 3 hours
- Write using a black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper.
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section.

Section II – 90 marks

- Attempt Questions 1-16
- Allow about 2 hours and 45 minutes for this section.

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1−10.

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1 Which of the following equations does NOT model a particle in simple harmonic motion with a period of $\frac{\pi}{2}$ seconds?

(A) $x(t) = \cos 4t + \cos\left(4t + \frac{2\pi}{3}\right)$

(B) $x(t) = 2 - 3\cos^2(2t)$

(C) $x(t) = \sin^4 t - \cos^4 t$

(D) $x(t) = \sin^6 t + \cos^6 t$

2 Which of the following is purely imaginary?

(A) $\left(1 - i\sqrt{3}\right)^6$

(B) $(1 + i)^{12}$

(C) $\left(\sqrt{3} + i\right)^8$

(D) None of the above.

3 A particle with mass 5 kilograms is projected to the right with an initial velocity of $10\,\mathrm{m\,s^{-1}}$, and it experiences a resistive force of $20v^2$, where v is the velocity of the particle after t seconds. Let a be the acceleration of the particle after t seconds.

Which of the following is the velocity-displacement equation of the particle?

(A) $v = 5e^{-2x}$

(B) $v = 10e^{-4x}$

(C) $v = 20e^{-2x}$

(D) $v = 40e^{-4x}$

4 Consider the following statement:

"To watch the live-stream, you must have an internet connection."

Which of the following is equivalent, in truth, to the converse statement?

(A) If you can watch the live stream then you have an internet connection.

(B) If you can't watch the live stream then you don't have an internet connection

(C) If you don't have an internet connection then you cannot watch the live stream.

(D) None of the above statements are equivalent.

5 Which of the following integrals has the largest value?

(A) $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^3}}\,dx$

(B) $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^4}}\,dx$

(C) $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{x-x^3}}\,dx$

(D) $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{x-x^4}}\,dx$

6 Let a and b be positive numbers. Which of the following is the least value of

$$\left(a+\frac{4}{b}\right)^2+\left(b+\frac{4}{a}\right)^2$$

(A) 4

(B) 16

(C) 32

(D) 50

7 Which of the following is a negation of the statement regarding non-zero integers a and b?

$$\forall a, b \in \mathbb{Z},\ \exists r \in \mathbb{Q},\ \frac{1}{a} < r < \frac{1}{b}$$

(A) For no such a and b does there exist a rational value r such that $\frac{1}{a} < r < \frac{1}{b}$

(B) For all a and b, there exists an irrational value r such that $\frac{1}{a} < r < \frac{1}{b}$

(C) There does not exist a rational r such that for all a and b, $\frac{1}{a} < r < \frac{1}{b}$

(D) There exists some rational r such that for no values of a and b, $\frac{1}{a} < r < \frac{1}{b}$

8 Consider the set of all complex numbers z satisfying the locus equation $|2z + i| = 1$.

Which of the following best describes the locus of $w = \dfrac{1}{z}$ on the Argand diagram?

(A) A horizontal line.

(B) A vertical line.

(C) A circle.

(D) A circle with an undefined point.

9 Consider $I_n = \displaystyle\int_0^{n\pi} \frac{\sin x}{1+x}\, dx$ for positive integer values of n.

Which of the following inequalities is true?

(A) $I_1 < I_2 < I_3 < I_4$

(B) $I_1 > I_2 > I_3 > I_4$

(C) $I_2 < I_4 < I_3 < I_1$

(D) $I_2 > I_4 > I_3 > I_1$

10 Suppose there is a game with three contestants where each contestant is wearing either a red or a blue hat. Everyone can see each others hat, but they cannot see their own hat.

At the end of each hour, if anyone has figured out the colour of their hat - they will announce their hat color to everyone, win a cash prize and exit the game.

The game starts, and someone from the crowd (who can see that all of the contestants is wearing a red hat) shouts out "I can see at least one red hat!".

What will happen to the contestants if this is the only information they receive and that their logic is always perfect?

(A) Nothing will happen.

(B) All of the contestants win after one hour.

(C) All of the contestants win after two hours.

(D) All of the contestants win after three hours.

Section II

90 marks
Attempt Questions 11–-16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available. In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (16 marks)

(a) Consider the two complex numbers $z = \cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}$ and $w = 1 + i\sqrt{3}$

(i) Express zw in the form $re^{i\theta}$. **2**

(ii) Hence find the second smallest positive integer value of n for which $(zw)^n$ is purely imaginary. **3**

(b) Find the equation of a sphere whose centre is at $(2, -1, 1)$ and touches the line with parametric equation given by **3**

$$\underset{\sim}{r} = \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

(c) Let ω be a non-real cube root of unity. Simplify **2**

$$\frac{(2 + 3\omega + 4\omega^2)(2 + \omega^2 + \omega^4)}{1 + 3\omega + 2\omega^2}$$

Question 11 continues on page 8

(d) (i) Prove that if x^2 is divisible by 6 for some integer x, then x is also divisible by 6. **2**

(ii) Use (i) to show that $\sqrt{6}$ is irrational. **1**

(iii) Hence, or otherwise, prove that $\sqrt{2} + \sqrt{3}$ is irrational. **1**

(e) A student researched the technique of "strong induction" and used it to attempt to prove that $2 \times n = 0$ for every non-negative integer n. The proof is as follows: **2**

> **Base case.** $n = 0$. Clearly $2 \times n = 0$.
> **Induction hypothesis.** Assume that the statement holds for all integers between 0 and k (inclusive).
>
> **Inductive step.** We aim to prove that, if the hypothesis is true, the statement is true for $n = k + 1$.
>
> Consider $k+1$ and write $k+1 = i+j$, where i and j are non-negative numbers. Then, by the hypothesis, we have that $2i = 2j = 0$ so:
>
> $$2(k + 1) = 2(i + j) = 2i + 2j = 0 + 0 = 0$$
>
> **Conclusion:** The base case is true. If the statement is true for the integers $0, 1, 2, \ldots, k$ then it is true for $k + 1$.
>
> So since it is true for $n = 0$, it is true for $n = 1$. Since it is true for $n = 0, 1$ then it is true for $n = 2$. Since it is true for $n = 0, 1, 2$ then it is true for $n = 3$ and so on.
>
> Hence, by mathematical induction, the statement is true for all non-negative integers.

Discuss the flaw with the students proof.

End of Question 11

Question 12 (15 marks)

(a) The complex number $3+i$ is a root of the polynomial $P(z) = z^4 - az^3 + bz^2 - az + 10$ where a and b are real numbers.

(i) Explain why $3 - i$ is also a zero of the polynomial $P(z)$. **1**

(ii) Find the remaining zeros of the polynomial $P(z)$ **3**

(b) A particle has acceleration equation **4**

$$\ddot{x} = -e^{-x} - e^{-2x},$$

and is initially at the origin with velocity $v = 2\,\mathrm{m\,s^{-1}}$.

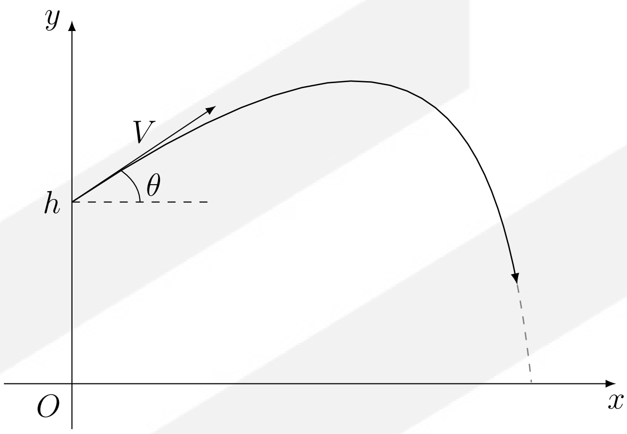
Show that $x = \ln(2e^t - 1)$.

(c) Find $\displaystyle\int_2^3 \frac{4}{(x-1)(x^2-1)}\,dx$. **3**

Question 12 continues on page 10

Question 12 (continued)

(d) A particle is thrown from the point O on the top of a very tall building of some height h with a fixed velocity V and variable angle θ subject to some resistance. A possible trajectory for a given h and θ is shown on the diagram below.



The equation of motion is

$$\underset{\sim}{a} = -k\underset{\sim}{v} - g\underset{\sim}{j}$$

where $\underset{\sim}{a}, \underset{\sim}{v}$ and $\underset{\sim}{r}$ are the acceleration, velocity and position vector respectively of the particle at time t. k is a positive constant corresponding to the magnitude of the resistance and g is the acceleration due to gravity.

(i) Prove that the horizontal displacement of the particle is given by **3**

$$x = \frac{V\cos\theta}{k}\left(1 - e^{-rt}\right)$$

where V and θ are the initial velocity and angle of projection respectively.

(ii) What is the closest distance to the base of the building that you could stand to ensure that you cannot be hit by the particle no matter the value of h? **1**

End of Question 12

Question 13 (15 marks)

(a) Consider the integral I_n given by

$$I_n = \int_{-\pi}^{\pi} \frac{\sin^2 \frac{nx}{2}}{\sin^2 \frac{x}{2}}\, dx$$

where n is a positive integer.

(i) Express $\cos((n+1)x) + \cos((n-1)x)$ in the form $2\cos A \cos B$. **1**

(ii) Hence, or otherwise, show that I_n forms an arithmetic progression. **3**

(iii) Hence, or otherwise, find I_{2024}. **1**

(b) Define the lines ℓ_1 and ℓ_2 by

$$\ell_1 : \quad \underset{\sim}{r}(\lambda) = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\ell_2 : \quad \underset{\sim}{r}(\mu) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

where $\underset{\sim}{v_1}$ and $\underset{\sim}{v_2}$ are the direction vectors of ℓ_1 and ℓ_2 shown above.

(i) Prove that ℓ_1 and ℓ_2 are neither parallel nor intersecting. **2**

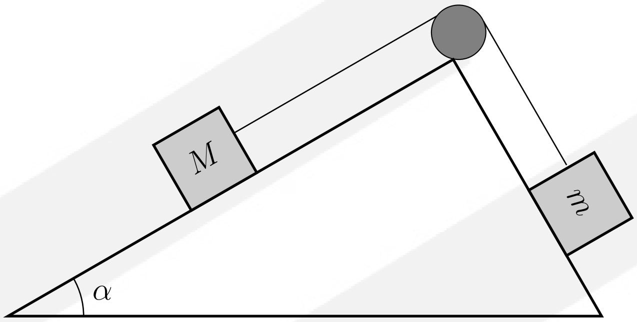
(ii) Find a vector $\underset{\sim}{w}$ that is perpendicular to both $\underset{\sim}{v_1}$ and $\underset{\sim}{v_2}$. **1**

(iii) Consider the points P and Q on ℓ_1 and ℓ_2 respectively such that $|\overrightarrow{PQ}|$ is minimised. **2**

Using the previous parts, or otherwise, find the shortest distance between ℓ_1 and ℓ_2.

Question 13 continues on page 12

(c) The diagram below shows two objects of mass M and m kilograms connected by a light inextensible string across a smooth pulley on a double-sided smooth right-angled ramp inclined at an angle of α.



(i) Show that if $\tan\alpha > \dfrac{m}{M}$ then particle A will slide down the ramp. **3**

(ii) Suppose $\tan\alpha = \dfrac{km}{M}$ where $k > 1$ such that particle A slides down the ramp. **2**

Find the value of k that maximises the acceleration.

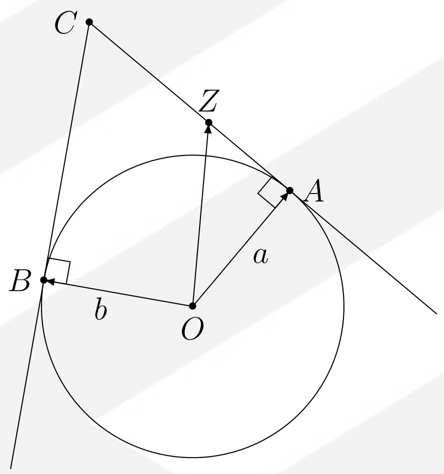
End of Question 13

Question 14 (16 marks)

(a) Let A and B be two points on the unit circle, centred at the origin, denoted by the complex numbers a and b respectively.

Let Z be a position vector on the tangent to the circle at the point A.

A tangent is also drawn at B which will intersect the other tangent at the point C as shown on the diagram below.



It can be shown that the radius of the circle is perpendicular to any tangent drawn to the circle. **(Do NOT prove this.)**

(i) Explain why $z = a(1 + ki)$ for some $k \in \mathbb{R}$. **1**

(ii) Prove that $z = 2a - a^2\bar{z}$ for all values of z on the tangent drawn to A. **3**

(iii) Show that $c = \dfrac{2ab}{a+b}$ and discuss what happens when $a = -b$ in relation to the diagram. **2**

(iv) Hence deduce the inequality $\left|\dfrac{1}{a} + \dfrac{1}{b}\right| < 2$ for any two distinct complex numbers with modulus 1 where $a + b \neq 0$. **1**

Question 14 continues on page 14

(b) Consider the statement for any square with a positive side length: "The area of a square is not equal to its perimeter."

(i) How many counter-examples are there to this statement? **1**

(ii) For the rest of this question consider the statement for rectangles with positive side lengths. **2**

The statement is now: "The area of a rectangle is not equal to its perimeter."

How many counter-examples are there to this statement?

(iii) Suppose the side lengths are restricted to the positive integers. **2**

List every counter-example to the statement in (ii) with this new restriction and prove that you have found them all.

(c) Consider $f(a) = a + \dfrac{1}{b(a-b)}$ which is defined for $a > b > 0$.

(i) Show that $f(a)$ is minimised when $a = b + \dfrac{1}{\sqrt{b}}$. **2**

(ii) Hence, or otherwise, deduce the minimum value of $a + \dfrac{1}{b(a-b)}$. **2**

End of Question 14

Question 15 (14 marks)

(a) Let $a_n = \text{Im}\left((1+2i)^n\right)$ be the imaginary part of the complex number $(1+2i)^n$ where n is a non-negative integer.

(i) Show that $a_{n+2} = 2a_{n+1} - 5a_n$ for $n \geq 0$. **2**

(ii) Prove that $a_n \neq 0$ for all $n \geq 1$. **2**

(iii) By considering the argument of $(1+2i)^n$, or otherwise, prove that $\tan^{-1}(2)$ cannot be expressed as a rational multiple of π. **2**

(b) Consider the sequences of real numbers $x_1, x_2, x_3, \ldots, x_n$ and $y_1, y_2, \cdots y_n$. **4**

Use mathematical induction to prove the *Cauchy-Schwarz* inequality:

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leq \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(b_1^2 + b_2^2 + \cdots + b_n^2\right)$$

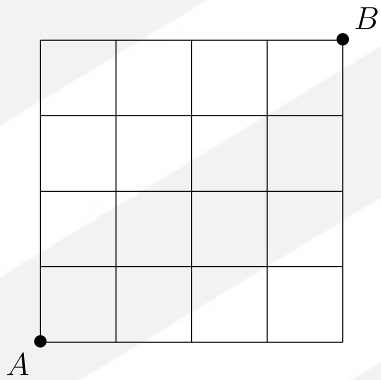
for integers $n \geq 2$.

Question 15 continues on page 16

(c) Consider a $n \times n$ grid with Alice and Bob beginning at opposite corners of the grid.

Alice will move towards the other corner by randomly choosing to step one unit up or to the right. Similarly Bob will move towards Alice's corner by moving one unit down or to the left.

The diagram below shows the case where $n = 4$.



(i) Show that the probability that Alice and Bob run into eachother is given by **2**

$$p_n = \frac{1}{2^{2n}}\left(\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2\right)$$

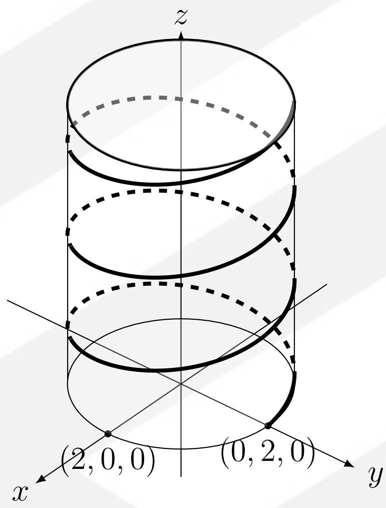
(ii) By considering the inequality in (b) or otherwise, deduce that $p_n \geq \dfrac{1}{n+1}$. **2**

End of Question 15

Question 16 (14 marks)

(a) Consider a cylinder of radius 2 and height 12 such that one of its circular faces is sitting on the xy-plane with centre at the origin.

Suppose some rope is uniformly wrapped around the cylinder starting at the point $(0,2,0)$ and ending at the point $(0,2,12)$ such that it is wrapped three times around the cylinder in a anti-clockwise orientiation as shown in the diagram below.



(i) Determine the vector equation for the position of the rope $\underset{\sim}{r}(t)$ where t satisfies the domain $0 \leq t \leq 12$. **2**

(ii) An ant is at the point $(2,0,12)$ and travels downwards along the rope. Meanwhile a second ant is at the point $(0,2,0)$ travels upwards along the rope at a different speed to the first. **3**

The first ant is slow and will travel the whole length of the rope in 10 seconds, whereas the second ant will travel the whole length in 1.25 seconds.

Both ants travel at constant (but different) speeds.

Determine the time and position at which the ants will collide.

Question 16 continues on page 18

(b) It is known that the statement $A \implies B$ is false only when A is true and B is false. **3**

Use a proof by contradiction, or otherwise, to show that the statement

$$((A \implies B) \text{ and } \neg B) \implies \neg B$$

is always true.

(Note: The notation $\neg A$ stands for the negation of A).

(c) The *home prime*, $f(n)$, of a integer $n \geq 2$ is obtained by the following list of steps:

- If n is prime, then $f(n) = n$.
- If n is not prime, then concatenate the prime factors of n in ascending order (with repeats allowed) to form the new number n^*, then $f(n) = f(n^*)$.

For example $9 = 3 \times 3$ has two prime factors so $f(9) = f(33)$. Then $33 = 3 \times 11$ so $f(33) = f(311)$. Then 311 is prime, so $f(311) = 311$ and the process terminates, so 311 is the home prime of the numbers $9, 33$ and 311.

Let $f_i(n)$ be the i^{th} iteration of this process for the calculation of $f(n)$. So in the example above we have

$$f_0(9) = 9, f_1(9) = 33, f_2(9) = 311 \text{ and } f_i(9) = 311 \text{ for all } i \geq 3.$$

(i) By considering mathematical induction on the number of prime factors, (possibly duplicates) of n, prove that $f_i(n)$ is a non-decreasing sequence. **4**

(ii) It is known that 832919 is prime. **(Do NOT prove this).** **2**

Solve the equation $f(n) = 832919$ for the possible values of n. Explain why you have found all possible solutions.

End of Question 16