Entwistle Mathematics

Year 12 Extension 1 Mock Exam

SOLUTIONS

General Instructions:

- Reading time – 10 minutes
- Working time – 2 hours
- Write using a black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper.
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section.

Section II – 60 marks

- Attempt Questions 1-14
- Allow about 1 hours and 45 minutes for this section.

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Which of the following is the solution to the inequality $\dfrac{1}{x-3} \geq 2$?

(A) $3 < x \leq \dfrac{7}{2}$

(B) $(x < 3) \cup \left(x > \dfrac{7}{2}\right)$

(C) $2 < x \leq 3$

(D) $(x < 2) \cup (x > 3)$

Solution:
(A)

$$\begin{aligned} \frac{1}{x-3} &\geq 2 \\ x-3 &\geq 2(x-3)^2 \\ (x-3)\,(2x-6-1) &\leq 0 \\ (x-3)(2x-7) &\leq 0 \end{aligned}$$

and so $3 < x \leq \dfrac{7}{2}$

2 Given three non-zero vectors $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$. If $\text{proj}_{\underset{\sim}{c}}\underset{\sim}{a} = \text{proj}_{\underset{\sim}{c}}\underset{\sim}{b}$ then which of the following is true?

(A) $\underset{\sim}{a} = \lambda \underset{\sim}{b}$ for some real constant λ

(B) $(\underset{\sim}{a} - \underset{\sim}{b}) \cdot \underset{\sim}{c} = 0$

(C) $\dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|} \geq 0$

(D) None of the above.

Solution:
(B): Substitute the formula for projection:

$$\frac{\underset{\sim}{a} \cdot \underset{\sim}{c}}{|\underset{\sim}{c}|^2}\underset{\sim}{c} = \frac{\underset{\sim}{b} \cdot \underset{\sim}{c}}{|\underset{\sim}{c}|^2}\underset{\sim}{c}$$
$$\underset{\sim}{a} \cdot \underset{\sim}{c} = \underset{\sim}{b} \cdot \underset{\sim}{c}$$
$$(\underset{\sim}{a} - \underset{\sim}{b}) \cdot \underset{\sim}{c} = 0$$

3 Which of the following is equal to $\int_0^{\frac{\pi}{4}} (1 + \tan x)^2 \, dx$?

(A) $1 + \ln 2$

(B) 1

(C) $1 - \ln 2$

(D) $\frac{\pi}{4} + \ln 2 + \frac{1}{3}$

Solution:
(A):

$$\begin{aligned}\int_0^{\frac{\pi}{4}} (1 + \tan x)^2 \, dx &= \int_0^{\frac{\pi}{4}} 1 + 2\tan x + \tan^2 x \, dx \\ &= \int_0^{\frac{\pi}{4}} 2\tan x + \sec^2 x \, dx \\ &= [\tan x - 2\ln|\cos x|]_0^{\frac{\pi}{4}} \\ &\dots \\ &= 1 + \ln 2\end{aligned}$$

4 How many solutions are there to the equation $\tan^{-1}(\tan x) + x = 0$?

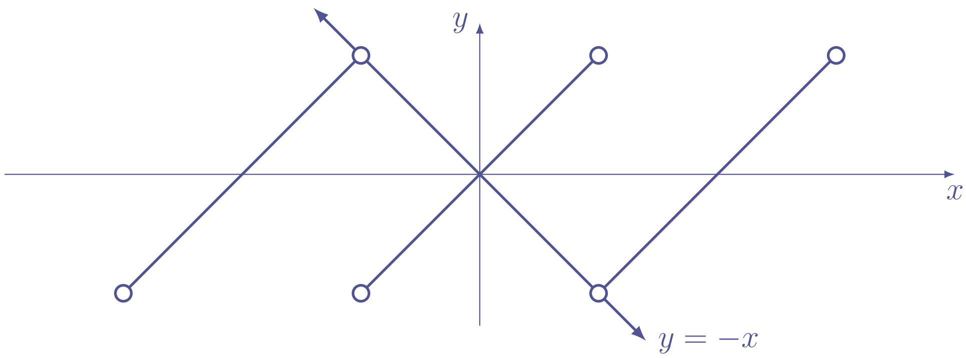
(A) 1

(B) 2

(C) 3

(D) None of the above.

Solution:
(A): Draw a quick sketch of $y = \tan^{-1}(\tan x)$ to see



Note that students who perform the following manipulation:

$$\begin{aligned}\tan^{-1}(\tan x) &= -x\\ \tan x &= \tan(-x)\\ \tan x &= -\tan x\\ 2\tan x &= 0\\ \tan x &= 0\\ \therefore x &= k\pi \text{ for } k \in \mathbb{Z}\end{aligned}$$

should be advised that this sort of algebra doesn't work in the inverse functions topic since there are restrictions to the domain and range that you should be keeping track of.

5 Which of the following is the value of $\cos^{-1}(\cos a)$ given that $\frac{3\pi}{2} < a < 2\pi$?

(A) $2\pi - a$

(B) $\pi - a$

(C) a

(D) $\pi + a$

Solution:
(A): a is in the fourth quadrant so we rewrite a as $2\pi - (2\pi - a)$ where $2\pi - a$ is now a first quadrant angle. Hence

$$\begin{aligned}\cos^{-1}(\cos a) &= \cos^{-1}(\cos(2\pi - a)) \\ &= 2\pi - a\end{aligned}$$

6 For which of the following values of n will

$$\left(x + \frac{1}{x^2}\right)^n$$

have a non-zero constant term?

(A) 2023

(B) 2024

(C) 2025

(D) 2026

Solution:
(C): The general $(k+1)^{\text{th}}$ term is of the form

$$T_{k+1} = \binom{n}{k} x^{n-k} x^{-2k} = \binom{n}{k} x^{n-3k}$$

where there is a constant term whenever $n = 3k$. So n needs to be a multiple of 3. Hence $n = 2025$ will work (and none of the other options).

7 Which of the following statements is not guaranteed to be always true?

(A) If polynomial $P(x)$, with degree greater than or equal to two, satisfies $P'(\alpha) = P(\alpha) = 0$ then α is at least a double root.

(B) A cubic polynomial with real coefficients either has one or three real roots.

(C) For a one-to-one function $f(x)$ with inverse $f^{-1}(x)$ where $f(a) = b$ for real numbers a and b, then $f^{-1}(b) = a$.

(D) If a function $f(x)$ intersects its inverse $f^{-1}(x)$ then it will do so on the line $y = x$.

Solution:
(D):

(A) is true - this is the double root theorem.

(B) is true.

(C) is true since it was specified that $f(x)$ was one-to-one.

(D) is not true. For example $y = -x^3$ intersects its inverse at the point $(-1, 1)$ which is not on the line $y = x$.

8 How many numbers must be selected (without replacement) from the set $\{1, 2, 3, 4, \ldots, 2n-1\}$ to guarantee that at least one pair of these numbers add up to $2n$?

(A) n

(B) $n - 1$

(C) $n + 1$

(D) $2n - 1$

Solution:
(A):

If we take 1, then take $2n - 2$. Take 3 then $2n - 4$ and so on.

If n is odd then we would need $n - 1$ pairs to guarantee this.

If n is even then we would need n pairs to guarantee this.

Since we don't know that n is odd or even, we would need to take n pairs to *guarantee* the property.

9 During the tea-break at a mathematics conference, 40% of attendees order tea and 60% order coffee (with no attendees ordering both drinks).

Of those who get tea, 85% add milk and of those who take coffee, 50% add milk.

Estimate the probability that, from a sample of 47 attendees, at least half of the attendees add milk to their drink.

(A) 0.9985

(B) 0.95

(C) 0.975

(D) 0.84

Solution:
(C):
Establish $P(T) = 0.4, P(C) = 0.6, P(T \cap C) = 0, P(M|T) = 0.85$ and $P(M|C) = 0.5$. Now

$$\begin{aligned} P(M) &= P(M|T)P(T) + P(M|C)P(C) \\ &= 0.85 \times 0.4 + 0.5 \times 0.6 \\ &= 0.64 \end{aligned}$$

So for $n = 47$, we have

$$P(\hat{p} \geq 0.5) = P\left(Z \geq \frac{0.5 - 0.64}{\sqrt{\dfrac{0.64 \times 0.36}{47}}} \right) = P(Z \geq -1.9995\ldots)$$

This is extremely close to -2 so we could just just empirical rule (the z table doesn't offer any better accuracy anyway) to obtain an estimate of 0.975

10 Consider a grid that consists of 9 cells within a 3×3 square.

A team of three players, who cannot see each-other, will **randomly** select a cell to draw a cross on any of the 9 squares. The three crosses are then drawn onto the board at the same time (with the possibility of there being multiple crosses on the same cell).

The game is won if the three crosses form a row, column or diagonal.

Which of the following is the probability that the game was "won"?

(A) $\dfrac{2}{21}$

(B) $\dfrac{34}{1701}$

(C) $\dfrac{8}{729}$

(D) $\dfrac{16}{243}$

Solution:
(D): There are 9^3 total ways for the players to select their cells.

There are 8 different rows, columns and diagonals. And 3! ways to permute the three players across them.

So $\dfrac{8 \times 3!}{9^3} = \dfrac{16}{243}$.

Section II

60 marks
Attempt Questions 11-–14
Allow about 1 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available. In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booket.

(a) The position of a projectile at time t is given by the following parametric equations: **2**

$$x = 5\sqrt{3}t$$
$$y = -5t^2 + 5t$$

Find the Cartesian equation of this trajectory in the from $y = ax^2 + bx$.

Solution:
We have that $t = \dfrac{x}{5\sqrt{3}}$ then substitute into y to obtain

$$y = -5\left(\frac{x^2}{25 \times 3}\right) + 5\left(\frac{x}{5\sqrt{3}}\right) = \frac{x}{\sqrt{3}} - \frac{x^2}{15}$$

(b) Consider the points P and Q with position vectors $\overrightarrow{OP} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ and $\overrightarrow{OQ} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$. **2**

If the point R divides the interval PQ internally in the ratio $1 : 2$, then find $\overrightarrow{PR}$ in the form $\begin{bmatrix} a \\ b \end{bmatrix}$.

Solution:
We have that $\overrightarrow{PQ} = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$ and $\overrightarrow{PR} = \frac{1}{3}\overrightarrow{PQ} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(c) Consider the vector $\underset{\sim}{u} = \underset{\sim}{i} + \sqrt{3}\,\underset{\sim}{j}$. **3**

Determine all the possible unit vector(s) $\underset{\sim}{v}$ where the acute angle between $\underset{\sim}{u}$ and $\underset{\sim}{v}$ is $\dfrac{\pi}{3}$.

Solution:
Let $\underset{\sim}{v} = \begin{bmatrix} a \\ b \end{bmatrix}$.

Then we have that $a + \sqrt{3}b = 2\cos\left(\frac{\pi}{3}\right) = 1$ as well as $a^2 + b^2 = 1$. Solve simultaneously to obtain

$$\begin{aligned}(1-\sqrt{3}b)^2 + b^2 &= 1\\ 1 - 2\sqrt{3} + 4b^2 &= 1\\ 4b^2 + 2\sqrt{3}b &= 0\\ b(2b+\sqrt{3}) &= 0\end{aligned}$$

so $b = 0$ (with $a = 1$) or $b = -\frac{\sqrt{3}}{2}$ with $a = \frac{5}{2}$).

(d) Determine the exact value of the integral $\int_0^{\frac{\pi}{2}} \sin(5x)\cos(x)\,dx$. **3**

Solution:

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin 5x \cos x\,dx &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(6x) + \sin(4x)\,dx\\ &= -\frac{1}{2}\left[\frac{\cos(6x)}{6} + \frac{\cos(4x)}{4}\right]_0^{\frac{\pi}{2}}\\ &\dots\\ &= \frac{1}{6}\end{aligned}$$

Question 11 continues on page 12

Question 11 (continued)

(e) Consider the polynomial $P(x) = x^3 + ax^2 + bx - 2$ where a and b are real numbers. **2**

It is given that $x^2 + 1$ is a factor of $P(x)$ and thus the following (incomplete) long division was determined:

$$\begin{array}{lllll} & & x & & \\ \hline x^3 + ax^2 & + & bx & - & 2 \\ x^3 & + & x & & \\ \hline & ax^2 + & (b-1)x & - & 2 \\ & & & & \\ \hline \end{array}$$

By completing the long division, find the values of a and b.

Solution:
Complete the division (multiply $x^2 + 1$ by a) to obtain

$$\begin{array}{lllll} & & x & + & a \\ \hline x^3 + ax^2 & + & bx & - & 2 \\ x^3 & + & x & & \\ \hline & ax^2 + & (b-1)x & - & 2 \\ & ax^2 & & + & a \\ \hline & & (b-1)x & - & (a+2) \end{array}$$

Since $x^2 + 1$ is a factor, the remainder $(b - 1)x - (a + 2)$ should be 0. Hence $b = 1$ and $a = -2$.

(f) Use a t-substitution to solve $2\sin\theta - \tan\theta = 6\cot\left(\frac{\theta}{2}\right)$ for $0 \leq \theta \leq 2\pi$. **3**

Solution:

Let $t = \tan\left(\frac{\theta}{2}\right)$ to obtain (on domain $0 \leq \frac{\theta}{2} \leq \pi$

$$\begin{aligned}
\frac{4t}{1+t^2} - \frac{2t}{1-t^2} &= \frac{6}{t} \\
\frac{2t}{1+t^2} - \frac{t}{1-t^2} &= \frac{3}{t} \\
2t^2(1-t^2) - t^2(1+t^2) &= 3(1-t^2)(1+t^2) \\
t^2 - 3t^4 &= 3 - 3t^4 \\
t^2 &= 3 \\
t &= \pm\sqrt{3} \\
\frac{\theta}{2} &= \frac{\pi}{3}, \frac{2\pi}{3} \\
\theta &= \frac{2\pi}{3}, \frac{4\pi}{3}
\end{aligned}$$

We must also check the value $\theta = \pi$:

$$\begin{aligned}
LHS &= 0 - 0 \\
RHS &= 0
\end{aligned}$$

so $\theta = \pi$ is also a solution.

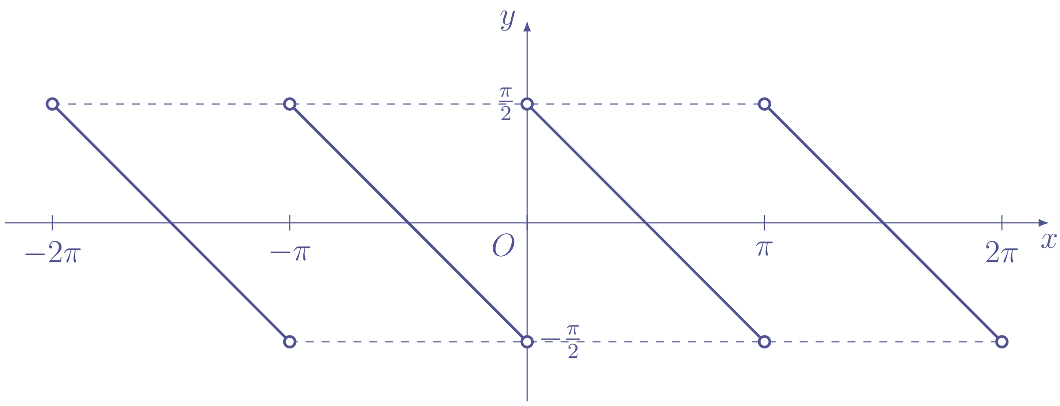
Hence our final solution is $\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$.

End of Question 11

Question 12 (17 marks) Use the Question 12 Writing Booket.

(a) Sketch the graph of $y = \tan^{-1}(\cot x)$ on $-2\pi \le x \le 2\pi$. **3**

Solution:



(b) By first expressing $\cos x - \sin x$ in the form $R\cos(x+\alpha)$, determine the maximum value of $2 + \cos x - \sin x$ and the smallest positive angle, in radians, at which this maximum value occurs. **4**

Solution:
Suppose $\cos x - \sin x = R\cos(x+\alpha) = R\cos\alpha\cos x - R\sin\alpha\sin x$ to conclude

$$R\cos\alpha = 1 \quad (1)$$
$$R\sin\alpha = 1 \quad (2)$$

$(1)^2 + (2)^2 \implies R^2 = 2 \implies R = \sqrt{2}$

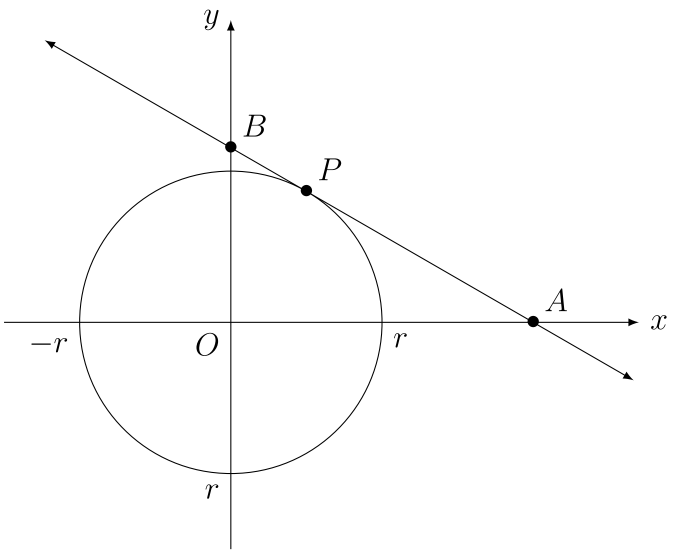
$(2) \div (1) \implies \tan\alpha = 1 \implies \alpha = \dfrac{\pi}{4}$.

Hence $2 + \cos x - \sin x = 2 + \sqrt{2}\cos\left(x + \dfrac{\pi}{4}\right)$ which has a maximum value of $2 + \sqrt{2}$ which occurs when $\cos\left(x + \dfrac{\pi}{4}\right) = 1$ so $x + \dfrac{\pi}{4} = 0, 2\pi, \ldots$ so $x = 2\pi - \dfrac{\pi}{4} = \dfrac{7\pi}{4}$.

Question 12 continues on page 15

Question 12 (continued)

(c) Consider the circle of radius r centred at the origin with a tangent line of the form $y = ax + b$ to the circle that cuts the x-axis at the points $A\left(-\frac{b}{a}, 0\right)$ and $B(0, b)$.



Let the projection of the vector $\overrightarrow{BO}$ onto the tangent be denoted as the vector $\overrightarrow{BP} = \underset{\sim}{p}$ for some point P.

(i) By considering the point(s) of intersection of the tangent to the circle, show that $r^2(a^2 + 1) = b^2$. **2**

(ii) Show that $\underset{\sim}{p} = -\frac{ar^2}{b}\begin{bmatrix}1\\a\end{bmatrix}$ and hence show that $|\overrightarrow{OP}| = r$. **2**

(iii) By taking an appropriate dot product, explain why the tangent is perpendicular to radius of the circle. **1**

Solution:

(i) $y = ax + b$ and $x^2 + y^2 = r^2$. So substitute the line into the circle (where we expect only one solution due to the line being a tangent):

$$x^2 + (a^2x^2 + 2abx + b^2) = r^2$$
$$(1 + a^2)x^2 + 2abx + (b^2 - r^2) = 0$$

Set the discriminant to 0:

$$4a^2b^2 - 4(1 + a^2)(b^2 - r^2) = 0$$
$$a^2b^2 - (1 + a^2)(b^2 - r^2) = 0$$
$$-b^2 + r^2 + a^2r^2 = 0$$
$$b^2 = r^2(a^2 + 1)$$

(ii):

$$\begin{aligned}\underset{\sim}{p} &= \text{Proj}_{\overrightarrow{BA}}\overrightarrow{BO} \\ &= \frac{-b^2}{\frac{b^2}{a^2}+b^2}\begin{bmatrix} b/a \\ b \end{bmatrix} \\ &= -\frac{ba}{1+a^2}\begin{bmatrix} 1 \\ a \end{bmatrix}\end{aligned}$$

but $1+a^2 = \dfrac{b^2}{r^2}$ from (i), so

$$\begin{aligned}\underset{\sim}{p} &= -\frac{ba}{\frac{b^2}{r^2}}\begin{bmatrix} 1 \\ a \end{bmatrix} \\ &= -\frac{ar^2}{b}\begin{bmatrix} 1 \\ a \end{bmatrix}\end{aligned}$$

where

$$\begin{aligned}|\overrightarrow{OP}|^2 &= b^2 - \frac{a^2r^4}{b^2}(1+a^2) \\ &= b^2 - \frac{a^2r^2\left[r^2(1+a^2)\right]}{b^2} \\ &= b^2 - a^2r^2 \\ &= r^2(1+a^2) - a^2r^2 \\ &= r^2\end{aligned}$$

so $|\overrightarrow{OP}| = r$.

(iii): Take $\underset{\sim}{p} \cdot \overrightarrow{BA}$ and confirm that we get 0. We have proven that the radius is perpendicular to the circle.

Question 12 continues on page 17

(d) Consider the differential equation $\dfrac{dy}{dx} = e^{x+y} + e^{x-y}$.

(i) Show that the differential equation is separable. **1**

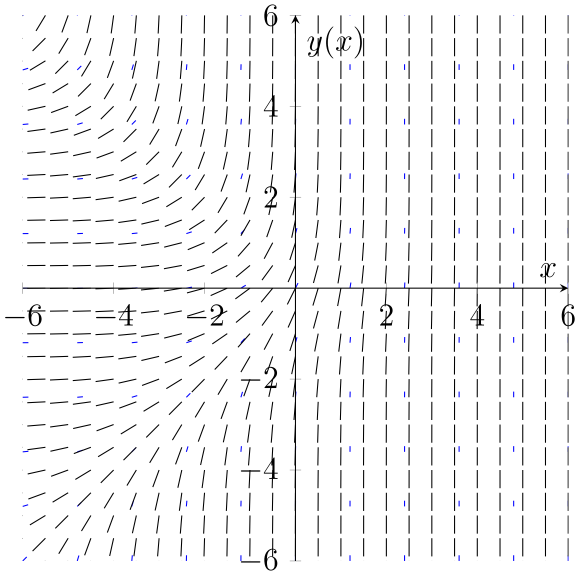
(ii) It is known that a particular solution curve has a horizontal asymptote along the x-axis as $x \to -\infty$. **3**

By solving the equation in (i), show that the particular solution is of the form

$$y = \ln(\tan(e^x + C))$$

and state the value of the constant C.

(iii) The diagram shows the direction field of the differential equation given. **1**



Use this to provide a sketch of the solution found in (ii), showing the equation of any asymptotes.

Solution:

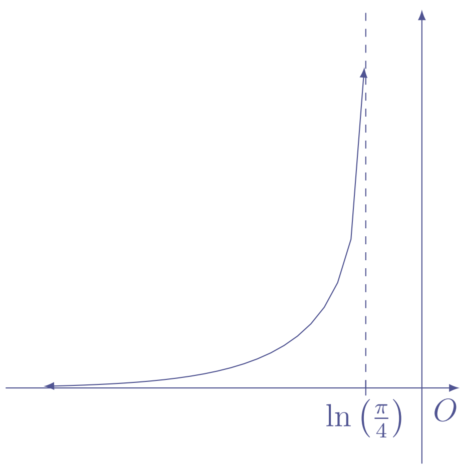
(i): Note that $\dfrac{dy}{dx} = e^x\left(e^y + e^{-y}\right)$ which is of the form $f(x)g(y)$.

(ii):

$$\int_0^y \frac{1}{e^y + e^{-y}}\,dy = \int_{-\infty}^x e^x\,dx$$
$$\int_0^y \frac{e^y}{1 + (e^y)^2}\,dy = [e^x]_0^x$$
$$\left[\tan^{-1}(e^y)\right]_0^y = e^x$$
$$\tan^{-1}(e^y) - \frac{\pi}{4} = e^x$$
$$\tan^{-1}(e^y) = e^x + \frac{\pi}{4}$$
$$e^y = \tan\left(e^x + \frac{\pi}{4}\right)$$
$$y = \ln\left(\tan\left(e^x + \frac{\pi}{4}\right)\right)$$

(iii): Note that there is a vertical asymptote at $e^x = \frac{\pi}{4} \implies x = \ln\left(\frac{\pi}{4}\right)$.

Then you can use the slope field for the rough sketch:

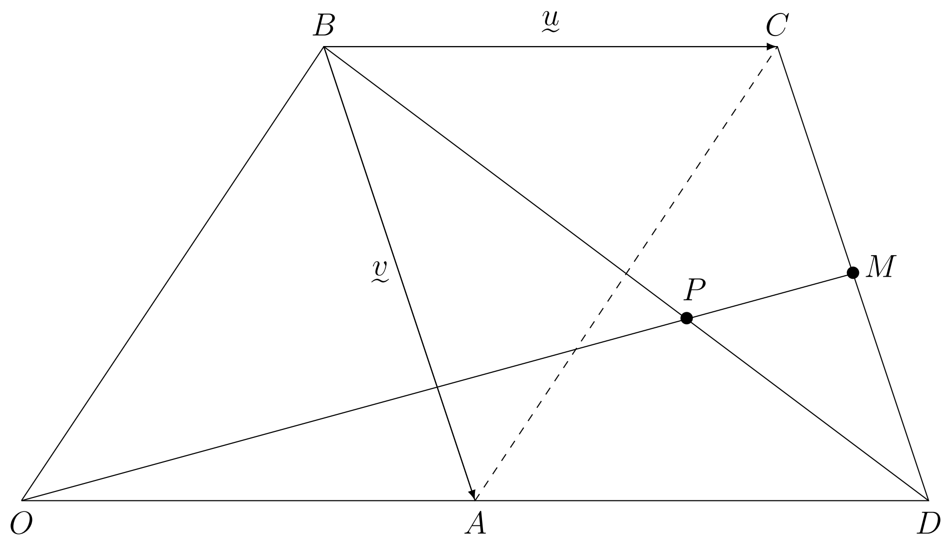


End of Question 12

Question 13 (12 marks) Use the Question 13 Writing Booket.

(a) Consider the parallelogram $ABCD$ where $\overrightarrow{BC} = \underset{\sim}{u}$ and $\overrightarrow{BA} = \underset{\sim}{v}$.

The line DA is extended to the point O such that OB is parallel to AC as shown in the diagram below.



Let M be the midpoint of CD and P be the intersection of the lines OM and BD.

(i) Determine $\overrightarrow{OM}$ in terms of $\underset{\sim}{u}$ and $\underset{\sim}{v}$. **2**

(ii) Suppose that P divides OM in the ratio $n : 1$. Find the value of n. **3**

Solution:

(i):

$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OB} + \overrightarrow{BC} + \frac{1}{2}\overrightarrow{CD} \\ &= (\underset{\sim}{u} - \underset{\sim}{v}) + \underset{\sim}{u} + \frac{1}{2}\underset{\sim}{v} \\ &= 2\underset{\sim}{u} - \frac{1}{2}\underset{\sim}{v}\end{aligned}$$

(ii): We have that $\overrightarrow{OP} = \frac{n}{n+1}\overrightarrow{OM} = \frac{n}{n+1}\left(2\underset{\sim}{u} - \frac{1}{2}\underset{\sim}{v}\right)$ but also $\overrightarrow{BP} = k\overrightarrow{BD} = k(\underset{\sim}{u} + \underset{\sim}{v})$ for some constant k. So $\overrightarrow{OP} = (\underset{\sim}{u} - \underset{\sim}{v}) + k(\underset{\sim}{u} + \underset{\sim}{v}) = (1+k)\underset{\sim}{u} + (k-1)\underset{\sim}{v}$. Now equate coefficients to obtain the equations:

$$\frac{2n}{n+1} = 1 + k \quad (1) \qquad \frac{n}{2(n+1)} = 1 - k \quad (2)$$

Then add (1) and (2) to obtain

$$\begin{aligned}\frac{2n}{n+1} + \frac{n}{2(n+1)} &= 2\\ \frac{n}{n+1}\left(2+\frac{1}{2}\right) &= 2\\ 5n &= 4(n+1)\\ \therefore n &= 4\end{aligned}$$

(b) Consider $f(x) = \tan^{-1}\left(\dfrac{x}{x+1}\right) + \tan^{-1}\left(\dfrac{1}{2x+1}\right)$.

(i) Show that $f'(x) = 0$ over the domain of $f(x)$. **2**

(ii) Hence, or otherwise, sketch the graph of $y = f(x)$. **2**

Solution:

$$\begin{aligned}f'(x) &= \frac{\frac{1\cdot(x+1)-x}{(x+1)^2}}{1+\frac{x^2}{(x+1)^2}} - \frac{\frac{2}{(2x+1)^2}}{1+\frac{1}{(2x+1)^2}}\\ &= \frac{1}{x^2+(x+1)^2} - \frac{2}{1+(2x+1)^2}\\ &= \frac{1}{2x^2+2x+1} - \frac{2}{4x^2+4x+2}\\ &= \frac{1}{2x^2+2x+1} - \frac{1}{2x^2+2x+1}\\ &= 0\end{aligned}$$

(ii): We conclude that $f(x)$ must be constant over the domain. Due to the discontinuities at $x=-1$ and $x=-\dfrac{1}{2}$ then $f(x)$ must be of the form

$$f(x) = \begin{cases} C_1 & x < -1\\ C_2 & -1 < x < -\frac{1}{2}\\ C_3 & = x > -\frac{1}{2}\end{cases}$$

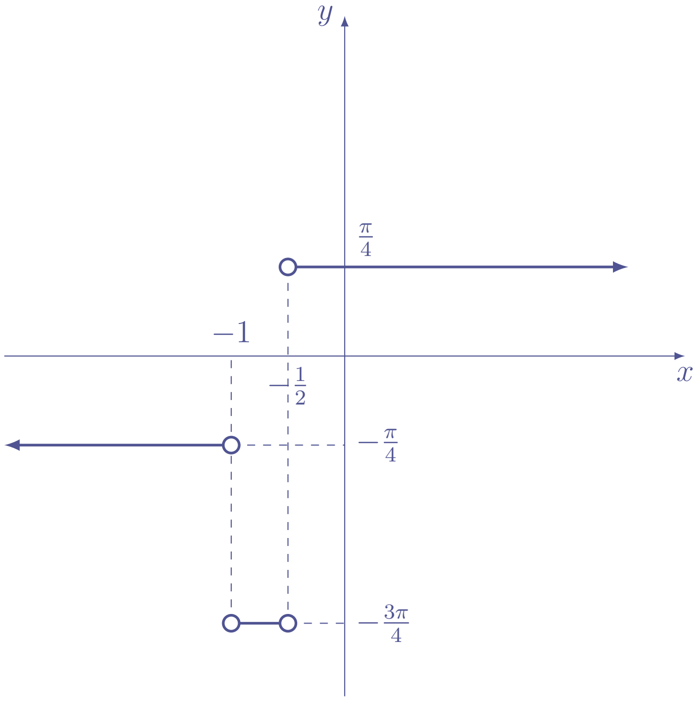
for (potentially different) constants C_1, C_2 and C_3.

As $x \to \infty$, $f(x) \to \dfrac{\pi}{4} + 0 = \dfrac{\pi}{4} = C_3$.

As $x \to -\infty$, $f(x) \to -\dfrac{\pi}{4} + 0 = -\dfrac{\pi}{4} = C_1$.

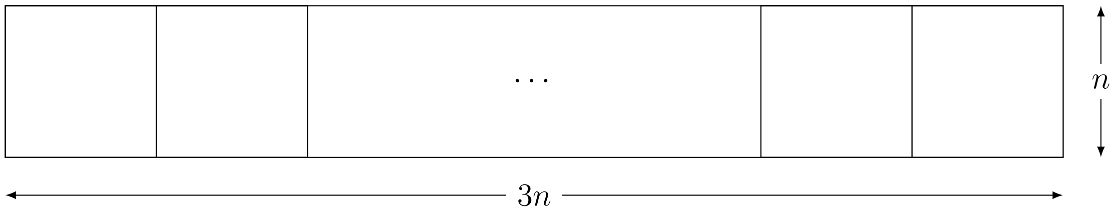
As $x \to -1^+$, $\dfrac{x}{x+1} \to -\infty$ and $\dfrac{1}{2x+1} \to -1$ so $f(x) \to -\dfrac{\pi}{2} - \dfrac{\pi}{4} = -\dfrac{3\pi}{4} = C_2$.

Hence the sketch is the following curve:



Question 13 continues on page 22

(c) A long rectangular strip of length $3n$ is divided into $3n$ $n \times n$ squares as shown below. **3**



Each square can be coloured red, blue or green.

Use a combinatorial argument to deduce the value of

$$\binom{3n}{0} \times 2^{3n} + \binom{3n}{1} \times 2^{3n-1} + \binom{3n}{2} \times 2^{3n-2} + \cdots + \binom{3n}{3n-1} \times 2 + \binom{3n}{3n}$$

(You **must** use a combinatorial argument - there will be no marks given for expanding binomials or using alternate methods).

Solution:
Condition on how many red squares are coloured.

If we use the color red r times, then we use blue and green (combined) $3n-r$ times. So choose which squares are coloured red in $\binom{3n}{r}$ ways, then there are two choices (blue or green) for each of the remaining $3n-r$ squares. Hence there are $\binom{3n}{r} 2^{3n-r}$ ways of using red r times.

Add all of the cases for $r = 0, 1, 2, \ldots, 3n$ to obtain the left hand side.

This clearly covers all of the possible ways of colouring the strip in total, so this will just evaluate to 3^{3n}.

End of Question 13

Question 14 (16 marks) Use the Question 14 Writing Booket.

(a) *You may use the z-score table that is provided for this question.*

A large company is holding a Christmas party and sends out invitations to its employees. All invited employees are guaranteed to come since it is during work hours and each employee is allowed to bring up to one "plus-one" (an additional guest who was not invited). **4**

Each employee will bring a plus-one with a probability of 40%.

Since the venue has a strict capacity of 400 guests due to space constraints, the company deliberately invites less employees in order to accommodate for any plus-ones.

Determine the maximum number of employees that the company should invite to ensure that there is less than a 1% chance of the venue exceeding capacity.

Solution:
We have that the number of attendees is $Y = X + n$ where X follows a binomial distribution with n trials and $p = 0.4$.
Note that $\mu = np = 0.4p$ and $\sigma = \sqrt{np(1-p)} = \sqrt{0.24p}$.

Now we want $P(n+X > 400) \leq 0.01$ so we make X the subject and then standardise to get the z-score:

$$\begin{aligned} P(n + X > 400) &\leq 0.01 \\ P(X > 400 - n) &\leq 0.01 \\ P\left(Z > \underbrace{\frac{400 - n - 0.4n}{\sqrt{0.24n}}}_{z}\right) &\leq 0.01 \end{aligned}$$

so, using z-tables, we require $z \geq 2.33$. Solve this inequality:

$$\begin{aligned} \frac{400 - 1.4n}{\sqrt{0.24n}} &\geq 2.33 \\ 400 - 1.4n &\geq 2.33\sqrt{0.24n} \\ 1.4n + 2.33\sqrt{0.24}\sqrt{n} - 400 &\leq 0 \\ 1.4u^2 + 2.33\sqrt{0.24}u - 400 &\leq 0 \end{aligned}$$

where we made the substitution $u = \sqrt{n}$. The equality is obtain when

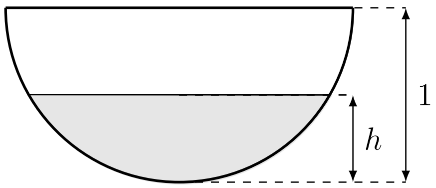
$$\begin{aligned} u &= \frac{-2.33\sqrt{0.24} \pm \sqrt{2.33^2 \times 0.24 + 4 \times 1.4 \times 400}}{2.8} \\ &= -17.316 \text{ and } 16.5003 \end{aligned}$$

where clearly $u \geq 0$. Hence we have

$$0 \leq \sqrt{n} \leq 16.5003 \Longrightarrow 0 \leq n \leq 272.2599\ldots.$$

Hence the maximum value of n is $n = 272$.

(b) A water tank has the shape of a horizontal cylinder with a radius of 1m and a length of 2m. The tank is currently empty and water is pumped into the tank at a rate of $\sqrt{h}$ litres per minute, where h is the variable depth of water filled in the cylinder at time t as shown in the cross section below.



(i) Show that the volume of water filled to a depth of h is given by **2**

$$V(h) = 4\int_{-1}^{h-1} \sqrt{1-y^2}\,dy$$

(ii) Use the substitution $y = \sin\theta$ to show that **3**

$$V(h) = \pi + 2\sin^{-1}(h-1) + 2(h-1)\sqrt{2h-h^2}$$

(iii) Find the volume of water in the tank after 1 minute. Leave your answer correct to three decimal places. **3**

Solution:

(i): We have that the area of the cross section is given by $A = 2\int_{-1}^{h-1} \sqrt{1-y^2}\,dy$ so

$$V = 2A = 4\int_{-1}^{h-1} \sqrt{1-y^2}\,dy.$$

(ii): Let $y = \sin\theta$ so $dy = \cos\theta\ d\theta$.

When $y = -1, \theta = -\dfrac{\pi}{2}$ and when $y = h-1, \theta = \sin^{-1}(h-1) := \alpha$. Put this into the integral:

$$\begin{aligned}
V &= 4\int_{-\frac{\pi}{2}}^{\alpha} \cos^2\theta\, d\theta \\
&= 2\int_{-\frac{\pi}{2}}^{\alpha} 1+\cos 2\theta\, d\theta \\
&= 2\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\alpha} \\
&= 2\left[\alpha + \frac{1}{2}\sin 2\alpha + \frac{\pi}{2}\right] \\
&= \pi + 2\alpha + \sin\alpha\cos\alpha \\
&= \pi + 2\sin^{-1}(h-1) + (h-1)\sqrt{1-(h-1)^2} \\
&= \pi + 2\sin^{-1}(h-1) + (h-1)\sqrt{2h-h^2}
\end{aligned}$$

(iii): We have that $\dfrac{dV}{dt} = \sqrt{h}$ so we should work out $\dfrac{dV}{dh}$. You could differentiate (ii) (but this would be very challenging). Instead we use the fact that

$$\frac{dV}{dh} = \frac{d}{dh}\left(4\int_{-1}^{h-1}\sqrt{1-y^2}\,dy\right) = 4\sqrt{1-(h-1)^2} = 4\sqrt{2h-h^2}$$

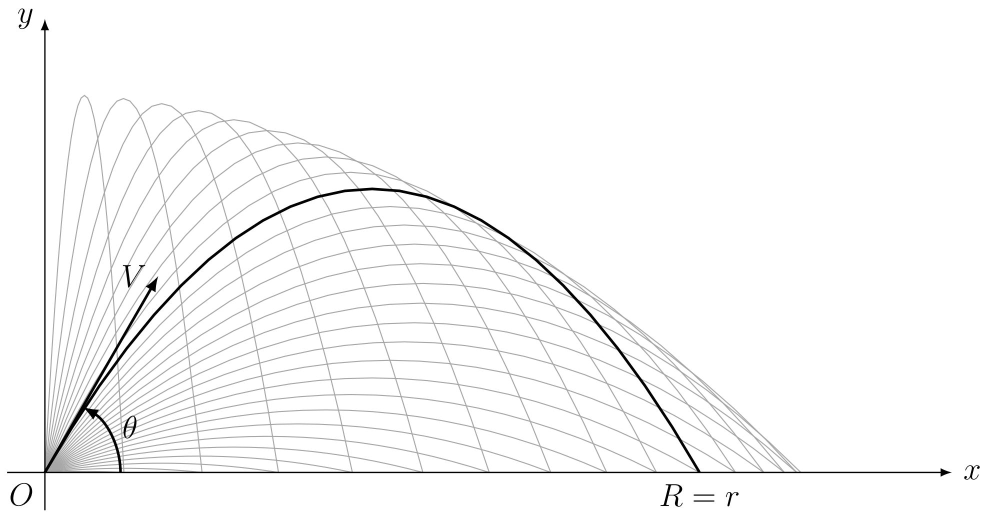
So now $\dfrac{dh}{dt} = \dfrac{dV}{dt} \times \dfrac{dh}{dV} = \sqrt{h} \times \dfrac{1}{4\sqrt{2h-h^2}} = \dfrac{1}{4\sqrt{2-h}}$. Solve this differential equation for the value of h at $t = 1$:

$$\begin{aligned}
\int_0^h \sqrt{2-h}\,dh &= \frac{1}{4}\int_0^1 dt \\
(2-h)^{\frac{3}{2}} - 2\sqrt{2} &= -\frac{3}{8} \\
2-h &= \left(\frac{16\sqrt{2}-3}{8}\right)^{\frac{2}{3}} \\
h &= 1 - \left(\frac{16\sqrt{2}-3}{8}\right)^{\frac{2}{3}}
\end{aligned}$$

Now substitute this into (ii) to obtain the volume $V \approx 0.282$.

Question 14 continues on page 26

(c) A projectile with a fixed velocity V is projected with a uniformly random angle θ $(0 < \theta < \frac{\pi}{2})$ to the horizontal which lands on the x-axis at some random variable R.



Given a gravitational acceleration of $g\,\mathrm{m\,s^{-2}}$, the displacement vector of a projectile with angle of projection θ and velocity $V\,\mathrm{m\,s^{-1}}$ can be shown to be

$$\underset{\sim}{r}(t) = \begin{bmatrix} Vt\cos\theta \\ Vt\sin\theta - \frac{1}{2}gt^2 \end{bmatrix} \qquad \textbf{(Do NOT prove this.)}$$

(i) The distribution of θ is given by the probability density function **1**

$$g(\theta) = \begin{cases} c & 0 < \theta < \frac{\pi}{2} \\ 0 & \text{else} \end{cases}$$

Determine the value of the constant c.

(ii) Show that the probability density function of R is given by **3**

$$f(r) = \begin{cases} \dfrac{2g}{\pi\sqrt{V^4 - r^2g^2}} & 0 < r < \dfrac{V^2}{g} \\ 0 & \text{else} \end{cases}$$

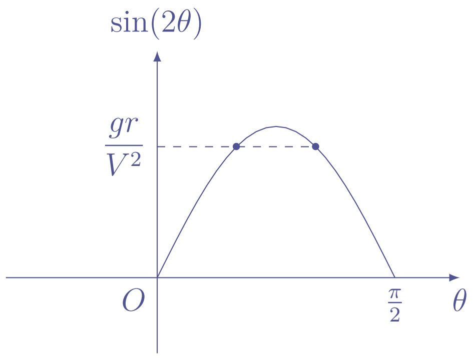
Solution:
(i): The area of the rectangle of height c and base $\frac{\pi}{2}$ needs to be 1 in order for $g(\theta)$ to be a probability density, so $c = \dfrac{2}{\pi}$.

(ii): The range for a projectile can be found by letting $y=0$ for t and then substitute t into x. This is an routine exercise and you should get $\dfrac{V^2\sin(2\theta)}{g}$.

Now we find the CDF of R, letting θ be a random variable:

$$\begin{aligned} F(r) &= P(R \leq r) \\ &= P\left(\frac{V^2\sin(2\theta)}{g} \leq r\right) \\ &= P\left(\sin(2\theta) \leq \frac{gr}{V^2}\right) \end{aligned}$$

Now consider the graph of $f(\theta) = \sin(2\theta)$:



We see that, by symmetry:

$$P\left(\sin(2\theta) \leq \frac{gr}{V^2}\right) = 2 \times P\left(2\theta \leq \sin^{-1}\left(\frac{gr}{V^2}\right)\right)$$

Hence

$$\begin{aligned} F(r) &= 2 \times P\left(2\theta \leq \sin^{-1}\left(\frac{gr}{V^2}\right)\right) \\ &= 2P\left(\theta \leq \underbrace{\frac{1}{2}\sin^{-1}\left(\frac{gr}{V^2}\right)}_{\alpha}\right) \\ &= 2\int_0^{\alpha} \frac{2}{\pi}\, d\theta \\ &= 2 \cdot \frac{2}{\pi}\alpha \\ &= \frac{2}{\pi}\sin^{-1}\left(\frac{gr}{V^2}\right) \end{aligned}$$

Finally we differentiate the CDF in order to obtain the PDF:

$$f(r) = \frac{2}{\pi}\frac{\frac{g}{V^2}}{\sqrt{1-\frac{g^2r^2}{V^4}}} = \frac{2g}{\pi V^2\sqrt{1-\frac{r^2g^2}{V^4}}} = \frac{2g}{\pi\sqrt{V^4-r^2g}}$$

where r ranges between 0 and the maximum range of $\dfrac{V^2}{2g}$.

End of Question 14